

ICE QUIVERS WITH POTENTIAL ARISING FROM ONCE-PUNCTURED POLYGONS AND COHEN-MACAULAY MODULES

LAURENT DEMONET AND XUEYU LUO

ABSTRACT. Given a tagged triangulation of a once-punctured polygon P^* with n vertices, we associate an ice quiver with potential such that the frozen part of the associated frozen Jacobian algebra has the structure of a Gorenstein $K[X]$ -order Λ . Then we show that the stable category of the category of Cohen-Macaulay Λ -modules is equivalent to the cluster category \mathcal{C} of type D_n . It gives a natural interpretation of the usual indexation of cluster tilting objects of \mathcal{C} by tagged triangulations of P^* . Moreover, it extends naturally the triangulated categorification by \mathcal{C} of the cluster algebra of type D_n to an exact categorification by adding coefficients corresponding to the sides of P . Finally, we lift the previous equivalence of categories to an equivalence between the stable category of graded Cohen-Macaulay Λ -modules and the bounded derived category of modules over a path algebra of type D_n .

1. INTRODUCTION

In a previous paper [10], we constructed ice quivers with potential arising from triangulations of polygons and we proved that the frozen parts of their frozen Jacobian algebras are orders. We proved that the categories of Cohen-Macaulay modules over these orders are stably equivalent to cluster categories of type A . The aim of this paper is to extend these results to tagged triangulations of once-punctured polygons to recover cluster categories of type D . We refer to [10] for a detailed introduction of the context and we will focus here on the tools we need specifically for this new case.

For every bordered surface with marked points, Fomin, Shapiro and Thurston introduced the concept of tagged triangulations and their mutations [12]. Then, they associated to each of these triangulations a quiver $Q(\sigma)$ and showed that the combinatorics of triangulations of the surface correspond to that of the cluster algebra defined by $Q(\sigma)$. Later in [25], Labardini-Fragoso associated a potential $W(\sigma)$ on $Q(\sigma)$. He proved that flips of triangulations are compatible with mutations of quivers with potential. This was generalized to the case of tagged triangulations by Labardini-Fragoso and Cerulli Irelli in [7, 26].

We refer to [3, 8, 30, 31] for a general background on Cohen-Macaulay modules (or lattices) over orders. Recently, strong connections between Cohen-Macaulay representation theory and tilting theory, especially cluster categories, have been established [1, 2, 9, 20, 21, 22, 24]. This paper enlarges some of these connections by dealing with frozen Jacobian algebras associated with tagged triangulations of once-punctured polygons from the viewpoint of Cohen-Macaulay representation theory.

Through this paper, let K denote a field and $R = K[X]$. We extend the construction of [12], and associate an ice quiver with potential (Q_σ, W_σ, F) to each tagged triangulation σ of a once-punctured polygon P^* with n vertices by adding a set F of n frozen vertices corresponding to the edges of the polygon and certain

arrows (see Definition 2.9). We study the associated frozen Jacobian algebra

$$\Gamma_\sigma := \mathcal{P}(Q_\sigma, W_\sigma, F)$$

(see Definition 2.1). Our main results are the following:

Theorem 1.1 (Theorem 2.19 and Theorem 2.30). *Let e_F be the sum of the idempotents of Γ_σ at frozen vertices. Then*

- (1) *the frozen Jacobian algebra Γ_σ has the structure of an R -order (see Definition 2.17 and Remark 2.18);*
- (2) *the frozen part $e_F \Gamma_\sigma e_F$ is isomorphic to the Gorenstein R -order*

$$\Lambda := \begin{bmatrix} R' & R' & R' & \cdots & R' & X^{-1}(X, Y) \\ (X, Y) & R' & R' & \cdots & R' & R' \\ (X) & (X, Y) & R' & \cdots & R' & R' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (X) & (X) & (X) & \cdots & R' & R' \\ (X) & (X) & (X) & \cdots & (X, Y) & R' \end{bmatrix}_{n \times n}, \quad (1.2)$$

where $R' = K[X, Y]/(Y(X - Y))$ and each entry of the matrix is a R' -submodule of $R'[X^{-1}]$.

Remark 1.3. In view of the isomorphism of R -algebras

$$\begin{aligned} R' &\cong R - R := \{(P, Q) \in R^2 \mid P - Q \in (X)\} \\ Y &\mapsto (0, X), \end{aligned}$$

we have an isomorphism

$$\Lambda \cong \begin{bmatrix} R - R & R - R & R - R & \cdots & R - R & R \times R \\ (X) \times (X) & R - R & R - R & \cdots & R - R & R - R \\ (X) - (X) & (X) \times (X) & R - R & \cdots & R - R & R - R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (X) - (X) & (X) - (X) & (X) - (X) & \cdots & R - R & R - R \\ (X) - (X) & (X) - (X) & (X) - (X) & \cdots & (X) \times (X) & R - R \end{bmatrix}_{n \times n}$$

(($(X) - (X)$ is the ideal of $R - R$ generated by (X, X)).

This order is part of a wide class of Gorenstein orders, called almost Bass orders, introduced and studied by Drozd-Kirichenko-Roiter and Hijikata-Nishida [16, 17] (see also [18]). More precisely, Λ is an *almost Bass order of type (III)*.

Theorem 1.4 (Theorems 2.30, 3.3, 3.16 and 3.19). *The category $\text{CM } \Lambda$ has the following properties:*

- (1) *For any tagged triangulation σ of P^* , we can map each tagged arc a of σ to the indecomposable Cohen-Macaulay Λ -module $e_F \Gamma_\sigma e_a$, where e_a is the idempotent of Γ_σ at a . This module does only depend on a (not on σ) and this map induces one-to-one correspondences*

$$\begin{aligned} \{\text{sides and tagged arcs of } P^*\} &\longleftrightarrow \{\text{indecomposable objects of } \text{CM } \Lambda\} / \cong \\ \{\text{sides of } P\} &\longleftrightarrow \{\text{indecomposable projectives of } \text{CM } \Lambda\} / \cong \\ \{\text{tagged triangulations of } P^*\} &\longleftrightarrow \{\text{basic cluster tilting objects of } \text{CM } \Lambda\} / \cong. \end{aligned}$$

- (2) *For the cluster tilting object $T_\sigma := e_F \Gamma_\sigma$ corresponding to a tagged triangulation σ ,*

$$\text{End}_{\text{CM } \Lambda}(T_\sigma) \cong \Gamma_\sigma^{\text{op}}.$$

- (3) *The category $\text{CM } \Lambda$ is 2-Calabi-Yau.*
- (4) *If K is a perfect field, there is a triangle-equivalence $\mathcal{C}(KQ) \cong \text{CM } \Lambda$, where Q is a quiver of type D_n and $\mathcal{C}(KQ)$ is the corresponding cluster category.*

Remark 1.5. To prove Theorem 1.4 (3), we establish that

$$\mathrm{CM} \Lambda \cong \mathrm{CM}^{\mathbb{Z}/n\mathbb{Z}} \left(\frac{K[x, y]}{(x^{n-1}y - y^2)} \right),$$

where x has degree 1 and y has degree -1 (modulo n).

Usually, the cluster category $\mathcal{C}(KQ)$ is constructed as an orbit category of the bounded derived category $\mathcal{D}^b(KQ)$. We can reinterpret this result in this context by studying the category of graded Cohen-Macaulay Λ -modules $\mathrm{CM}^{\mathbb{Z}} \Lambda$:

Theorem 1.6 (Theorem 4.5). *Using the same notation as before:*

- (1) *The Cohen-Macaulay Λ -module $e_F \Gamma_\sigma$ can be lifted to a tilting object in $\mathrm{CM}^{\mathbb{Z}}(\Lambda)$.*
- (2) *There exists a triangle-equivalence $\mathcal{D}^b(KQ) \cong \mathrm{CM}^{\mathbb{Z}}(\Lambda)$.*

In Section 2, we introduce ice quivers with potential (Q_σ, W_σ, F) associated with tagged triangulations σ of a once-punctured polygon P^* . We also introduce combinatorial and algebraic elementary tools in subsection 2.3. Finally, we prove in this section that the frozen Jacobian algebra Γ_σ associated with (Q_σ, W_σ, F) is an R -order, as well as $\Lambda \cong e_F \Gamma_\sigma e_F$ which is independent of σ . In section 3, we classify Cohen-Macaulay modules over Λ , we compute homological properties of $\mathrm{CM} \Lambda$ and we establish the correspondence between tagged triangulations of P^* and basic cluster tilting objects of $\mathrm{CM} \Lambda$. Thus, after proving that $\mathrm{CM} \Lambda$ is Frobenius stably 2-Calabi-Yau, we conclude that $\mathrm{CM} \Lambda$ is stably triangle equivalent to a cluster category of type D . In section 4, we deal with results about $\mathrm{CM}^{\mathbb{Z}} \Lambda$.

Notice that the naive generalization of these results to other surfaces do not hold in general as shown in Subsection 2.5 for a digon with two punctures.

Acknowledgements We would like to show our gratitude to the second author's supervisor, Osamu Iyama, for his guidance and valuable discussions. We also thank him for introducing us this problem and showing us enlightening examples.

2. ICE QUIVERS WITH POTENTIAL ASSOCIATED WITH TRIANGULATIONS

In this section, we introduce ice quivers with potential associated with tagged triangulations of a once-punctured polygon and their frozen Jacobian algebras. We show that in any case, the frozen Jacobian algebra has the structure of an R -order, and its frozen part is isomorphic to a given R -order Λ defined in (1.2).

2.1. Frozen Jacobian algebras. We refer to [11] for background about quivers with potential. Let Q be a finite connected quiver without loops, with set of vertices $Q_0 = \{1, \dots, n\}$ and set of arrows Q_1 . As usual, if $\alpha \in Q_1$, we denote by $s(\alpha)$ its starting vertex and by $e(\alpha)$ its ending vertex. We denote by KQ_i the K -vector space with basis Q_i consisting of paths of length i in Q , and by $KQ_{i, \mathrm{cyc}}$ the subspace of KQ_i spanned by all cycles in KQ_i . Consider the path algebra $KQ = \bigoplus_{i \geq 0} KQ_i$. An element $W \in \bigoplus_{i \geq 1} KQ_{i, \mathrm{cyc}}$ is called a *potential*. Two potentials W and W' are called *cyclically equivalent* if $W - W'$ belongs to $[KQ, KQ]$, the vector space spanned by commutators. A *quiver with potential* is a pair (Q, W) consisting of a quiver Q without loops and a potential W which does not have two cyclically equivalent terms.

For each arrow $\alpha \in Q_1$, the cyclic derivative ∂_α is the linear map $\bigoplus_{i \geq 1} KQ_{i, \mathrm{cyc}} \rightarrow KQ$ defined on cycles by

$$\partial_\alpha(\alpha_1 \cdots \alpha_d) = \sum_{\alpha_i = \alpha} \alpha_{i+1} \cdots \alpha_d \alpha_1 \cdots \alpha_{i-1}.$$

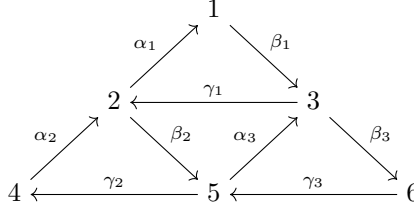


FIGURE 2.2. Example of iced quiver with potential

Definition 2.1 ([5]). An *ice quiver with potential* is a triple (Q, W, F) , where (Q, W) is a quiver with potential and F is a subset of Q_0 . Vertices in F are called *frozen vertices*.

The *frozen Jacobian algebra* is defined by

$$\mathcal{P}(Q, W, F) = KQ / \mathcal{J}(W, F),$$

where $\mathcal{J}(W, F)$ is the ideal

$$\mathcal{J}(W, F) = \langle \partial_\alpha W \mid \alpha \in Q_1, s(\alpha) \notin F \text{ or } e(\alpha) \notin F \rangle$$

of KQ .

Example 2.3. Consider the quiver Q of Figure 2.2 with potential $W = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 - \gamma_1\beta_2\alpha_3$ and set of frozen vertices $F = \{4, 5, 6\}$. Then the Jacobian ideal is

$$\mathcal{J}(W, F) = \langle \beta_1\gamma_1, \gamma_1\alpha_1, \alpha_1\beta_1 - \beta_2\alpha_3, \beta_2\gamma_2, \gamma_2\alpha_2 - \alpha_3\gamma_1, \beta_3\gamma_3 - \gamma_1\beta_2, \gamma_3\alpha_3 \rangle.$$

Note that this ice quiver with potential appeared from preprojective algebras [5, 13].

2.2. Ice quivers with potential arising from triangulations. We recall the definition of triangulations of a polygon with one puncture and introduce our definition of ice quivers with potential arising from tagged triangulations of a polygon with one puncture.

Definition 2.4. Let P be a regular polygon with n vertices and n sides. Fix a marked point in the center of P . Then the marked point is called a *puncture* and the combination of P and the puncture is called a *(once-)punctured polygon* P^* . We call *interior of P^** for the interior of the polygon P excluding the puncture. We denote by M the set of the n vertices of the polygon and the puncture.

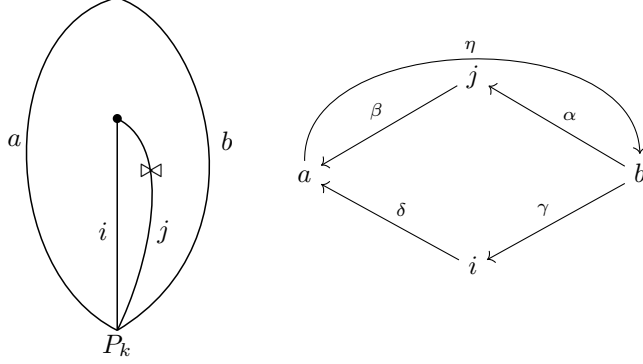
Definition 2.5 (Tagged arcs). [12, Definition 7.1] A *tagged arc* in the punctured polygon P^* is a curve a in P such that

- (1) the endpoints of a are distinct in M ;
- (2) a does not intersect itself;
- (3) except for the endpoints, a is disjoint from M and from the sides of P ;
- (4) a does not cut out an unpunctured digon. (In other words, a is not contractible onto the sides of P .)

Each arc a is considered up to isotopy inside the class of such curves.

Moreover, each arc incident to the puncture has to be tagged either *plain* either *notched*.

In the figures, the plain tags are omitted while the notched tags are represented by the symbol \bowtie .



We depict the part of Q_σ quiver induced by a once-punctured digon. Notice that there are two other arrows linking a and b if P is the digon itself.

FIGURE 2.10. Once-punctured digon

Definition 2.6 (Compatibility of tagged arcs, [12, Definition 7.4]). Two tagged arcs a and b are *compatible* if and only if the following conditions are satisfied:

- (1) there are curves in their respective isotopy classes whose relative interiors do not intersect;
- (2) if a and b are incident to the puncture and not isotopic, they are either both plain, either both notched.

Definition 2.7 ([12]). A *tagged triangulation* of the punctured polygon P^* is the union of the set of the sides of P and any maximal collection of pairwise compatible tagged arcs of P^* .

Remark 2.8. The set of all tagged arcs in a punctured polygon is finite. Moreover, any tagged triangulation can be realized up to isotopy as a collection of tagged non-intersecting arcs.

Let us now define ice quivers with potential arising from punctured polygons:

Definition 2.9. Let P^* be a punctured polygon with n sides and σ be a tagged triangulation of P^* . For convenience, the n sides of P and all the tagged arcs of σ are called the edges of σ . A *true triangle* of σ is a triangle consisting of edges of σ such that the puncture is not in its interior.

We assign to σ two ice quivers with potential (Q_σ, W_σ, F) and $(Q'_\sigma, W'_\sigma, F)$ as follows.

The quiver Q'_σ is a quiver whose vertices are indexed by the edges of σ . Whenever two edges a and b are sides of a common true triangle of σ , then Q'_σ contains an internal arrow $a \rightarrow b$ in the true triangle if a is a predecessor of b with respect to anticlockwise orientation centred at the joint vertex. For every vertex of the polygon P , there is an external arrow $a \rightarrow b$ where a and b are its two incident sides of P , a being a predecessor of b with respect to anticlockwise orientation centred at the joint vertex. Moreover, if the puncture is adjacent to exactly one notched arc and one plain arc of σ , we have the configuration shown in Figure 2.10.

Then, the quiver Q_σ is obtained from Q'_σ by removing external arrows winding around vertices of P with no incident tagged arc in σ .

We say that a cycle of Q_σ (resp. Q'_σ) is *planar* if it does not contain any arrow of Q_σ (resp. Q'_σ) in its interior and each arrow appears at most once. Notice that for the definition of planar, the quivers are not abstract but embedded in the plane (each internal arrow being drawn inside the triangle it is constructed from, and each

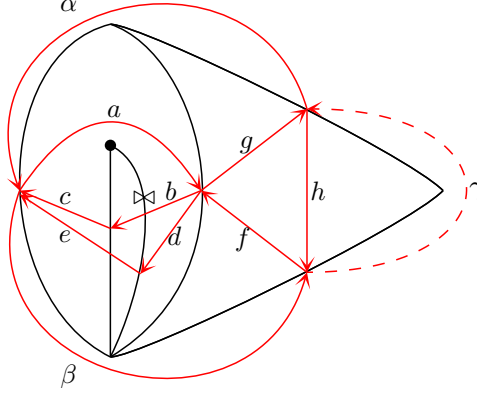


FIGURE 2.12. Quivers associated with a tagged triangulation

external arrow winding around the corresponding vertex outside the polygon). We have the following possible different kinds of planar cycles in Q_σ and Q'_σ :

- (1) *clockwise triangles* which come from true triangles in σ ;
- (2) an *anticlockwise punctured cycle* which consists of the arrows connecting arcs incident to the puncture.
- (3) *anticlockwise external cycles* which contain exactly one external arrow and each of which is centered at a vertex of P .

We define F as the subset of $(Q_\sigma)_0$ indexed by the n sides of the polygon P . The potential W_σ (resp. W'_σ) is defined as

$$\begin{aligned} & \sum \text{clockwise triangles} - \sum \text{anticlockwise external cycles} \\ & - \text{the anticlockwise punctured cycle} \end{aligned}$$

in Q_σ (resp. Q'_σ).

When there is a once-punctured digon in the triangulation σ as shown in Figure 2.10, we have to adapt slightly the previous definition. The anticlockwise external cycle centred at P_k which is taken in account is the one containing $\gamma\delta$. On the other hand, both $\eta\alpha\beta$ and $\eta\gamma\delta$ appear as clockwise triangles in W_σ and W'_σ . In this case, there is no anticlockwise punctured cycle. An explicit case involving a once-punctured digon is described in Proof of Lemma 2.27.

Example 2.11. Let us consider the triangulation σ of Figure 2.12. We drew the corresponding quivers (γ is in Q'_σ but not in Q_σ). We have

$$W_\sigma = fgh + abc + ade - \alpha ag - \beta fbc \quad \text{and} \quad W'_\sigma = W_\sigma - \gamma h.$$

From now on, for any tagged triangulation σ of the punctured polygon P^* , denote $\mathcal{P}(Q_\sigma, W_\sigma, F)$ by Γ_σ .

Remark 2.13. (1) We can also realize Γ_σ as

$$\Gamma_\sigma \cong \frac{KQ'_\sigma}{\mathcal{J}'(W'_\sigma)} \quad \text{where} \quad \mathcal{J}'(W'_\sigma) := \langle \partial_\alpha W'_\sigma \mid \alpha \in Q_1, \alpha \text{ is not external} \rangle.$$

We will use both definitions freely depending on convenience.

- (2) Notice that (Q'_σ, W'_σ) and (Q_σ, W_σ) are not necessarily reduced, in the sense that oriented 2-cycles can appear in the potential, because some vertices of the polygon have no incident tagged arcs in σ , or because the puncture

has exactly two non-isotopic incident tagged arcs. Thus, it is possible that non-admissible relations appear.

- (3) All arrows of Q_σ (resp. Q'_σ) either appear once in W_σ (resp. W'_σ), either twice with opposite signs. Thus all relations derived from the potential are either commutativity relations (of the form $w = w'$ for two paths w and w' of length at least 1) or 0 relations (of the form $w = 0$ for a path w of length at least 1).

2.3. Notation and preliminaries. The vertices of the polygon P are called P_1, P_2, \dots, P_n counter-clockwisely. When we do computations on the indices of vertices of P , we compute modulo n . If $r, s \in \llbracket 1, n \rrbracket := \{1, 2, \dots, n\}$, we denote

$$d(r, s) := \begin{cases} s - r, & \text{if } s \geq r, \\ s - r + n, & \text{if } s < r. \end{cases}$$

We also denote $\llbracket r, s \rrbracket = \{r, r+1, \dots, s\}$, $\llbracket r, s \llbracket = \llbracket r, s \rrbracket \setminus \{s\}$, $\rrbracket r, s \rrbracket = \llbracket r, s \rrbracket \setminus \{r\}$ and $\llbracket r, s \rrbracket = \llbracket 1, n \rrbracket \setminus \llbracket s, r \rrbracket$ (notice that $\llbracket r, r \rrbracket = \llbracket 1, n \rrbracket \setminus \{r\}$). If A is a condition, we define δ_A to be 1 if A is satisfied and 0 if A is not satisfied.

For $r, s, t \in \llbracket 1, n \rrbracket$, we will use freely the identities $\llbracket r, s \rrbracket = \llbracket r+1, s+1 \rrbracket$, $\llbracket r, s \rrbracket = \llbracket 1, n \rrbracket \setminus \llbracket s, r \rrbracket$, $\llbracket r, s \llbracket = \llbracket 1, n \rrbracket \setminus \llbracket s, r \rrbracket$ and $\delta_{r \in \llbracket s, t \rrbracket} = \delta_{s \in \llbracket t, r \rrbracket} = \delta_{t \in \llbracket r, s \rrbracket}$.

If $r, s \in \llbracket 1, n \rrbracket$, we denote by (P_r, P_s) the arc going from P_r to P_s turning counter-clockwisely around the puncture (thus, (P_r, P_{r+1}) is a side of the polygon and (P_{r+1}, P_r) is not except if $n = 2$). We denote by $(P_r, *)$ the plain tagged arc from P_r to the puncture and by (P_r, \bowtie) the notched tagged arc from P_r to the puncture.

From now on, we always denote $a = (P_{a_1}, P_{a_2})$ if a is not incident to the puncture and $a = (P_{a_1}, *)$ or $a = (P_{a_1}, \bowtie)$ if a is incident to the puncture (in the latter case, we fix by convention $a_2 = a_1$). We need to fix some geometrical definition. To make it precise, we suppose that P is a regular polygon inscribed in the unit circle and that the puncture is at the origin of the plane. Then, \vec{a} is the vector from P_{a_1} to P_{a_2} if $a_1 \neq a_2$ and it is the unit vector tangent at P_{a_1} to the unit circle in the clockwise direction if $a_1 = a_2$.

If a and b are tagged arcs of P^* , we define

$$\ell_{a,b}^\theta := d(a_1, b_1) + d(a_2, b_2) + n \left| \delta_{a_1 \in \llbracket b_1, a_2 \rrbracket} - \delta_{b_2 \in \llbracket b_1, a_2 \rrbracket} \right|.$$

Lemmas 2.14 and 2.15 are elementary observations about ℓ^θ . Proofs are computational and given for the sake of completeness. We suggest the reader to skip them first.

Lemma 2.14. *If a and b are two sides or tagged arcs of P^* , the angle from \vec{a} to \vec{b} is*

$$\frac{\pi}{n} \ell_{a,b}^\theta,$$

up to a multiple of 2π .

Proof. First of all, in complex coordinates, if $a_1 \neq a_2$,

$$\begin{aligned} \vec{a} &= \exp\left(2\pi i \frac{a_2}{n}\right) - \exp\left(2\pi i \frac{a_1}{n}\right) \\ &= \exp\left(\pi i \frac{a_1 + a_2}{n}\right) \left(\exp\left(\pi i \frac{a_2 - a_1}{n}\right) - \exp\left(\pi i \frac{a_1 - a_2}{n}\right) \right) \\ &= \exp\left(\pi i \frac{a_1 + a_2}{n}\right) 2i \sin\left(\pi \frac{a_2 - a_1}{n}\right) \end{aligned}$$

so the argument of \vec{a} is

$$\frac{\pi}{n} \left(a_1 + a_2 + \frac{n}{2} + n \delta_{a_1 \geq a_2} \right)$$

(note that this formula works also if $a_1 = a_2$). So the angle from \vec{a} to \vec{b} is

$$\frac{\pi}{n} (b_1 - a_1 + b_2 - a_2 + n (\delta_{b_1 \geq b_2} - \delta_{a_1 \geq a_2})).$$

As $\ell_{a,b}^\theta$ is clearly invariant by rotation of the polygon, as well as the angle from \vec{a} to \vec{b} , we can suppose that $a_2 = 1$ and the angle from a to b becomes

$$\begin{aligned} & \frac{\pi}{n} (d(a_1, b_1) - n\delta_{a_1 > b_1} + d(1, b_2) + n (\delta_{b_1 \geq b_2} - 1)) \\ &= \frac{\pi}{n} (d(a_1, b_1) + d(1, b_2) - n\delta_{a_1 > b_1} - n\delta_{b_1 < b_2}) \\ &= \frac{\pi}{n} (d(a_1, b_1) + d(1, b_2) - n\delta_{a_1 \in]b_1, 1[} - n\delta_{b_2 \in]b_1, 1[}) \end{aligned}$$

which is clearly congruent to $\pi\ell_{a,b}^\theta/n$ modulo 2π . \square

Another important point is that ℓ^θ is subadditive:

Lemma 2.15. *If a, b and c are three sides or tagged arcs of P^* , then $\ell_{a,b}^\theta + \ell_{b,c}^\theta \geq \ell_{a,c}^\theta$. More precisely,*

- if a is a side of P ,

$$\ell_{a,b}^\theta + \ell_{b,c}^\theta = \ell_{a,c}^\theta + 2n (\delta_{c_2 \in]c_1, b_2[} \delta_{b_1 \in]b_2, c_1[} + \delta_{a_1 \in]b_1, c_1[});$$

- if b is a side of P ,

$$\ell_{a,b}^\theta + \ell_{b,c}^\theta = \ell_{a,c}^\theta + 2n (\delta_{a_1 \in]a_2 - 1, c_1[} \delta_{c_2 \in]a_2 - 1, c_1[} + \delta_{b_1 \in]c_1, a_2 - 1[});$$

- if c is a side of P ,

$$\ell_{a,b}^\theta + \ell_{b,c}^\theta = \ell_{a,c}^\theta + 2n (\delta_{a_1 \in]b_1, a_2[} \delta_{b_2 \in]a_2, b_1[} + \delta_{c_2 \in]a_2, b_2[}).$$

Proof. We have:

$$\begin{aligned} & \ell_{a,b}^\theta + \ell_{b,c}^\theta - \ell_{a,c}^\theta \\ &= d(a_1, b_1) + d(b_1, c_1) - d(a_1, c_1) + d(a_2, b_2) + d(b_2, c_2) - d(a_2, c_2) \\ & \quad + n (|\delta_{a_1 \in]b_1, a_2[} - \delta_{b_2 \in]b_1, a_2[}| + |\delta_{b_1 \in]c_1, b_2[} - \delta_{c_2 \in]c_1, b_2[}| \\ & \quad - |\delta_{a_1 \in]c_1, a_2[} - \delta_{c_2 \in]c_1, a_2[}|) \\ &= n (\delta_{b_1 \in]c_1, a_1[} + \delta_{b_2 \in]c_2, a_2[} + |\delta_{a_1 \in]b_1, a_2[} - \delta_{b_2 \in]b_1, a_2[}| \\ & \quad + |\delta_{b_1 \in]c_1, b_2[} - \delta_{c_2 \in]c_1, b_2[}| - |\delta_{a_1 \in]c_1, a_2[} - \delta_{c_2 \in]c_1, a_2[}|). \end{aligned}$$

If this quantity was negative, we would have $b_1 \in]a_1, c_1[$ and $b_2 \in]a_2, c_2[$ and:

- either $a_1 \in]c_1, a_2[$ and $c_2 \in]a_2, c_1[$. As $c_2 \notin]c_1, b_2[$, we get $b_1 \notin]c_1, b_2[$ so $b_1 \in]b_2, c_1[$. It is then easy to deduce that $a_1 \in]b_1, a_2[$ and $b_2 \notin]b_1, a_2[$ and it contradicts the hypothesis.
- either $a_1 \in]a_2, c_1[$ and $c_2 \in]c_1, a_2[$. As $a_1 \notin]b_1, a_2[$, we get $b_2 \notin]b_1, a_2[$ so $b_2 \in]a_2, b_1[$. It is then easy to deduce that $b_1 \notin]c_1, b_2[$ and $c_2 \in]c_1, b_2[$ and it contradicts the hypothesis.

In any case, we reached a contradiction.

Notice that for any four $i, j, k, l \in]1, n[$, we have the identities

$$\begin{aligned} |\delta_{i \in]k, j[} - \delta_{l \in]k, j[}| &= \delta_{i \in]k, j[} \delta_{l \in]j, k[} + \delta_{i \in]j, k[} \delta_{l \in]k, j[} \\ &= \delta_{i \in]k, j[} \delta_{l \in]j, k[} + (1 - \delta_{i \in]k, j[})(1 - \delta_{l \in]j, k[}) \\ &= 2\delta_{i \in]k, j[} \delta_{l \in]j, k[} + 1 - \delta_{i \in]k, j[} - \delta_{l \in]j, k[} \end{aligned}$$

$$\begin{aligned} \text{and } |\delta_{i \in]k, j[} - \delta_{l \in]k, j[}| &= \delta_{i \in]k, j[} \delta_{l \in]j, k[} + \delta_{i \in]j, k[} \delta_{l \in]k, j[} \\ &= \delta_{i \in]k, j[} (1 - \delta_{l \in]k, j[}) + (1 - \delta_{i \in]k, j[}) \delta_{l \in]k, j[} \\ &= \delta_{i \in]k, j[} + \delta_{l \in]k, j[} - 2\delta_{i \in]k, j[} \delta_{l \in]k, j[}. \end{aligned}$$

If a is a side of the polygon, we have $a_2 = a_1 + 1$ and the previous difference becomes (up to a factor n):

$$\begin{aligned}
& \delta_{b_1 \in \llbracket c_1, a_1 \rrbracket} + \delta_{b_2 \in \llbracket c_2, a_1+1 \rrbracket} + |\delta_{a_1 \neq b_1} - \delta_{b_2 \in \llbracket b_1, a_1+1 \rrbracket}| \\
& + |\delta_{b_1 \in \llbracket c_1, b_2 \rrbracket} - \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket}| - |\delta_{a_1 \neq c_1} - \delta_{c_2 \in \llbracket c_1, a_1+1 \rrbracket}| \\
& = \delta_{a_1 \in \llbracket b_1, c_1 \rrbracket} + \delta_{b_2 \in \llbracket c_2, a_1+1 \rrbracket} + \delta_{a_1 \neq b_1} - \delta_{b_2 \in \llbracket b_1, a_1+1 \rrbracket} \\
& + 2\delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} + 1 - \delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} - \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} - \delta_{a_1 \neq c_1} + \delta_{c_2 \in \llbracket c_1, a_1+1 \rrbracket} \\
& = \delta_{a_1 \in \llbracket b_1, c_1 \rrbracket} + 2\delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} + 1 - \delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} - \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} \\
& + \delta_{a_1 \in \llbracket b_2-1, c_2-1 \rrbracket} - \delta_{a_1=b_1} - \delta_{a_1 \in \llbracket b_2-1, b_1-1 \rrbracket} + \delta_{a_1=c_1} + \delta_{a_1 \in \llbracket c_2-1, c_1-1 \rrbracket} \\
& = \delta_{a_1 \in \llbracket b_1, c_1 \rrbracket} + 2\delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} - \delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} - \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} \\
& + \delta_{a_1 \in \llbracket b_2-1, c_2-1 \rrbracket} - \delta_{a_1=b_1} + \delta_{a_1 \in \llbracket b_1-1, b_2-1 \rrbracket} + \delta_{b_1=b_2} + \delta_{a_1=c_1} + \delta_{a_1 \in \llbracket c_2-1, c_1-1 \rrbracket} \\
& = \delta_{a_1 \in \llbracket b_1, c_1 \rrbracket} + 2\delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} - \delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} - \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} \\
& + \delta_{a_1 \in \llbracket b_1-1, c_1-1 \rrbracket} + \delta_{b_2 \in \llbracket c_1, b_1 \rrbracket} + \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket} - \delta_{a_1=b_1} + \delta_{a_1=c_1} \\
& = 2\delta_{a_1 \in \llbracket b_1, c_1 \rrbracket} + 2\delta_{b_1 \in \llbracket b_2, c_1 \rrbracket} \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket}.
\end{aligned}$$

The other computations are analogous. \square

Let us introduce the K -algebras that will play an important role in this paper.

As before, $R = K[X]$. We define the R -algebra $R' = K[X, Y]/(YX - Y^2)$. It is in fact an R -order of rank 2 (see Definition 2.17), and we have the following R -isomorphism with a classical Bass order:

$$R' \rightarrow R - R := \{(P, Q) \in R^2 \mid P - Q \in (X)\}, \quad Y \mapsto (0, X).$$

The three indecomposable Cohen-Macaulay R' -modules and irreducible morphisms over R appear in each of the two lines of the following commutative diagram:

$$\begin{array}{ccccc}
(Y) & \xleftarrow{\quad \iota \quad} & R' & \xleftarrow{\quad \pi \quad} & R'/(Y) \\
\uparrow \wr & \swarrow Y & \parallel & \searrow X-Y & \downarrow \wr \\
R'/(X-Y) & \xleftarrow{\quad Y \quad} & R' & \xleftarrow{\quad X-Y \quad} & (X-Y) \\
& \nwarrow \pi' & & \nearrow \iota' &
\end{array}$$

where Y and $X - Y$ are multiplications, π are projections and ι are natural inclusions.

Finally, we denote by \mathcal{R}' the algebra $K[u^{\pm 1}, v]/(vu - v^2)$ where R' is seen as a subalgebra of \mathcal{R}' through the inclusion

$$R' \hookrightarrow \mathcal{R}', \quad X \mapsto u^{2n}, \quad Y \mapsto v^{2n}. \quad (2.16)$$

2.4. Frozen Jacobian algebras are R -orders. Let (Q_σ, W_σ, F) be an ice quiver with potential arising from a tagged triangulation σ as defined in Section 2.2 and e_i be the trivial path of length 0 at vertex i . The main result of this section is that $\Gamma_\sigma := \mathcal{P}(Q_\sigma, W_\sigma, F)$ (Definition 2.1) is an R -order.

First, we introduce the definition of orders and Cohen-Macaulay modules over orders.

Definition 2.17. Let S be a commutative Noetherian ring of Krull dimension 1. An S -algebra A is called an S -order if it is a finitely generated S -module and $\text{soc}_S A = 0$. For an S -order A , a left A -module M is called a (maximal) *Cohen-Macaulay A -module* if it is finitely generated as an S -module and $\text{soc}_S M = 0$ (or equivalently $\text{soc}_A M = 0$). We denote by $\text{CM } A$ the category of Cohen-Macaulay A -modules. It is a full exact subcategory of $\text{mod } A$.

Remark 2.18. If S is a principal ideal domain (e.g. $S = R$) and $M \in \text{mod } S$, then $\text{soc}_S M = 0$ if and only if M is free as an S -module.

We refer to [3],[8],[30] and [31] for more details about orders and Cohen-Macaulay modules.

The main theorem of this subsection is the following.

Theorem 2.19. *The frozen Jacobian algebra Γ_σ has the structure of an R -order.*

The remaining part of this subsection is devoted to prove Theorem 2.19. The strategy is to define a grading on Γ_σ , to prove that the centre $Z(\Gamma_\sigma)$ of Γ_σ is R' and to give its order structure as an R' -module. Notice that the center of Jacobian algebras coming from surfaces without boundary was computed by Ladkani in [27, Proposition 4.11].

We describe Γ_σ in details in Proposition 2.26. Let us define a grading on Q_σ (and Q'_σ).

Definition 2.20 (θ -length). Let a and b be sides or tagged arcs of P^* and $\alpha : a \rightarrow b$ be an arrow of Q'_σ . Let θ be the value of the oriented angle from \vec{a} to \vec{b} taken in $[0, 2\pi)$. We define the θ -length of α by

$$\ell^\theta(\alpha) = \frac{n}{\pi}\theta.$$

The θ -length of arrows extends additively to a map ℓ^θ from paths to integers, defining a grading on KQ_σ (and KQ'_σ) which will also be denoted by ℓ^θ .

Remark 2.21. Using Lemma 2.14, we see easily that for any arrow $\alpha : a \rightarrow b$, $\ell^\theta(\alpha) = \ell^\theta_{a,b}$. Indeed, if a and b share a common endpoint, then $0 \leq \ell^\theta_{a,b} < 2n$.

We now prove that for any tagged arcs or sides a and b of P^* , the possible θ -lengths of paths from a to b in Q_σ does not depend on the triangulation σ containing a and b .

Proposition 2.22. *Let σ and σ' be two different triangulations of the punctured polygon P^* . For any two edges a and b common to σ and σ' , the minimal θ -length of paths from a to b in Q_σ is the same as the one in $Q_{\sigma'}$.*

Proof. Any two triangulations can be related by a sequence of flips such that each time we only change one arc in the related triangulation to get another one. Therefore, without losing generality, we can assume that the two triangulations σ and σ' have the same arcs except one. We show the possible differences between σ and σ' in Figure 2.23.

It is sufficient to prove that for any two vertices common to both triangulations, and for any path between them in one triangulation, we can find a path with the same θ -length in the other triangulation. In each case, certain compositions of two arrows in one of the diagrams have to be replaced by one arrow in the other one. We can check case by case that the θ -length of both coincide.

For example, suppose that the triangulations only differ in a square not incident to the puncture as shown at the top of Figure 2.23. Considering the given position of the puncture, the arcs are (P_i, P_k) , (P_i, P_j) , (P_i, P_l) , (P_j, P_k) , (P_l, P_j) and (P_l, P_k) . And we get that

$$\begin{aligned} & \ell^\theta((P_j, P_k), (P_i, P_j)) + \ell^\theta((P_i, P_j), (P_l, P_j)) \\ &= d(j, i) + d(k, j) + d(i, l) = d(j, l) + d(k, j) = \ell^\theta((P_j, P_k), (P_l, P_j)) \end{aligned}$$

and the other cases work in the same way. \square

Proposition 2.24. *The potential W_σ (resp. W'_σ) on Q_σ (resp. Q'_σ) is homogeneous of θ -length $2n$. Thus, ℓ^θ induces a grading on Γ_σ .*

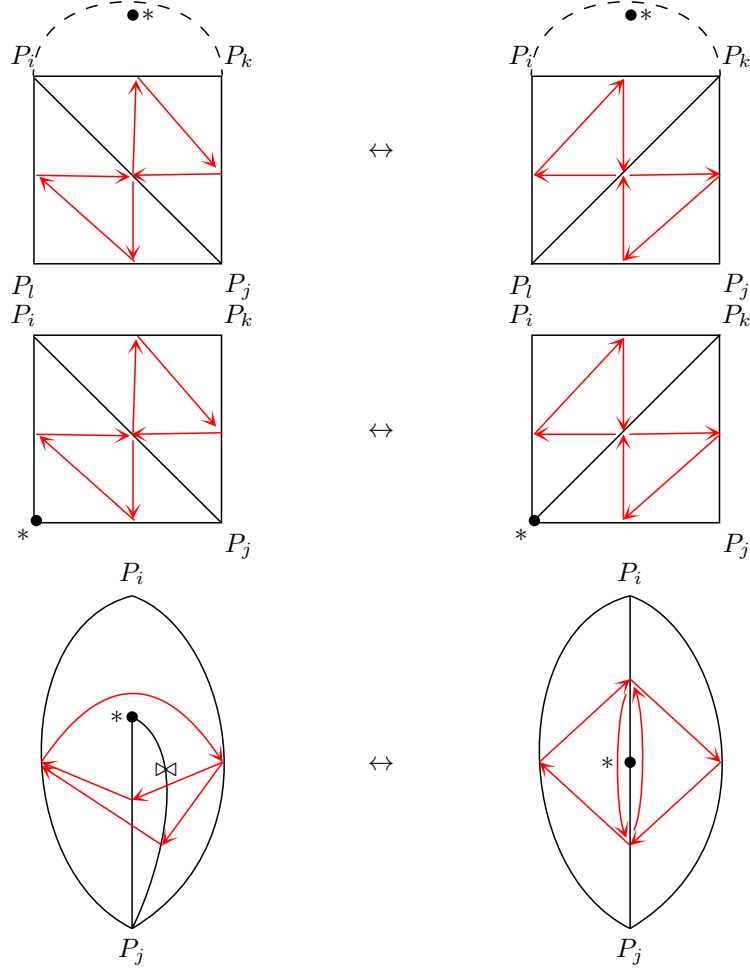


FIGURE 2.23. Possible flips

Proof. Consider a true triangle of σ . Up to cyclic permutation, we can denote its three sides by a, b, c in clockwise order, satisfying $a_1 = b_2, a_2 = c_2$. Moreover, they satisfy either $b_1 = c_1$ and $a_1 \in \llbracket b_1, a_2 \rrbracket$ (if the triangle is not incident to the puncture) either $b_1 = b_2$ and $c_1 = c_2$ (if the triangle is incident to the puncture). In any case, the θ -length of the clockwise triangle induced by this true triangle is

$$\begin{aligned}
 & \ell_{a,b}^\theta + \ell_{b,c}^\theta + \ell_{c,a}^\theta \\
 &= (d(a_1, b_1) + d(b_1, c_1) + d(c_1, a_1)) + (d(a_2, b_2) + d(b_2, c_2) + d(c_2, a_2)) \\
 & \quad + n (|\delta_{a_1 \in \llbracket b_1, a_2 \rrbracket} - \delta_{b_2 \in \llbracket b_1, a_2 \rrbracket}| + |\delta_{b_1 \in \llbracket c_1, b_2 \rrbracket} - \delta_{c_2 \in \llbracket c_1, b_2 \rrbracket}| \\
 & \quad + |\delta_{c_1 \in \llbracket a_1, c_2 \rrbracket} - \delta_{a_2 \in \llbracket a_1, c_2 \rrbracket}|) = n + n + n \times 0 = 2n.
 \end{aligned}$$

Using flips introduced in the proof of Proposition 2.22, we can transform σ to a triangulation τ that consists of the sides of P and the tagged arcs incident to the puncture (as in Figure 2.29, page 16). Moreover, using the reasoning of Proposition 2.22, flips clearly conserve the θ -length of anticlockwise planar cycles of Q'_σ winding around the puncture or vertices of P^* . Therefore, it is enough to see that W'_τ is homogeneous of θ -length $2n$. This is easy to check by calculation. As terms of W_σ are terms of W'_σ , W_σ is also homogeneous. \square

Proposition 2.25. *Let a and b be two edges of σ which are not incident to the puncture or tagged in the same way. The minimal θ -length of paths from a to b in Q_σ is $\ell_{a,b}^\theta$.*

Proof. Let us prove first that there exists a path from a to b with θ -length $\ell_{a,b}^\theta$. Let us do an induction on $\ell_{a,b}^\theta$. If it is 0, then $a = b$ and the result is obvious. If a and b are both incident to the puncture and not isotopic, the result is immediate (consider the triangulation τ consisting the sides of P and the plain tagged arcs incident to the puncture).

Let us suppose that a is not incident to the puncture. We consider four cases:

- Suppose that $b_1, b_2 \neq a_2$. Let us consider an arc c such that $c_1 = a_1$ and $c_2 = a_2 + 1$ and tagged in the same way as b if $b_1 = b_2$ and $c_1 = c_2$. The arc c is either isotopic to the union of a and a side of the polygon (if $a_2 \neq a_1 - 1$), either to a part of a (if $a_2 = a_1 - 1$). In any case, a , b and c are compatible so we can choose a triangulation σ' containing a , b and c . In $Q_{\sigma'}$, there is an arrow α from a to c of θ -length 1, and, as

$$\begin{aligned} \ell_{c,b}^\theta &= d(c_1, b_1) + d(c_2, b_2) + n |\delta_{c_1 \in \llbracket b_1, c_2 \rrbracket} - \delta_{b_2 \in \llbracket b_1, c_2 \rrbracket}| \\ &= d(a_1, b_1) + d(a_2 + 1, b_2) + n |\delta_{a_1 \in \llbracket b_1, a_2 + 1 \rrbracket} - \delta_{b_2 \in \llbracket b_1, a_2 + 1 \rrbracket}| \\ &= d(a_1, b_1) + d(a_2, b_2) - 1 + n |\delta_{a_1 \in \llbracket b_1, a_2 \rrbracket} - \delta_{b_2 \in \llbracket b_1, a_2 \rrbracket}| = \ell_{a,b}^\theta - 1 \end{aligned}$$

we can apply the induction hypothesis: there is a path w from c to b of θ -length $\ell_{c,b}^\theta$ and the path αw has the expected θ -length $\ell_{a,b}^\theta$. Finally, thanks to Proposition 2.22, there is a path of the same θ -length in σ .

- Suppose that $b_1, b_2 \neq a_1$ and $a_2 \neq a_1 + 1$. The same reasoning works with $c_1 = a_1 + 1$ and $c_2 = a_2$.
- Suppose that $b_1 = a_2$ and $b_2 = a_1$. In this case, we construct σ' by putting a , b and the two tagged arcs connecting the puncture and a_2 . In σ' , there is an arrow from a to b which has, by definition, θ -length $\ell_{a,b}^\theta$.
- Suppose that $a_2 = a_1 + 1$ and $b_1 \neq a_1$. In this case, put $c = (P_{a_1+1}, P_{a_2+1})$ and complete to a triangulation σ' (containing b). We have an arrow α from a to c of θ -length 2. Moreover,

$$\begin{aligned} \ell_{c,b}^\theta &= d(c_1, b_1) + d(c_2, b_2) + n |\delta_{c_1 \in \llbracket b_1, c_2 \rrbracket} - \delta_{b_2 \in \llbracket b_1, c_2 \rrbracket}| \\ &= d(a_1 + 1, b_1) + d(a_2 + 1, b_2) + n |\delta_{a_1+1 \in \llbracket b_1, a_2+1 \rrbracket} - \delta_{b_2 \in \llbracket b_1, a_2+1 \rrbracket}| \\ &= d(a_1, b_1) + d(a_2, b_2) - 2 + n |\delta_{a_1 \in \llbracket b_1, a_2 \rrbracket} - \delta_{b_2 \in \llbracket b_1, a_2 \rrbracket}| = \ell_{a,b}^\theta - 2 \end{aligned}$$

and the same reasoning as before works.

The case where b is not incident to the puncture is similar.

Let us now prove by induction on the θ -length of any path w from a to b that this θ -length is at least $\ell_{a,b}^\theta$. When $a = b$ or if w is an arrow, it is clear. When a and b are different and w is not an arrow, w is the composition of two nonzero paths w' from a to c and w'' from c to b . By induction hypothesis and Lemma 2.15, we have

$$\ell^\theta(w) = \ell^\theta(w') + \ell^\theta(w'') \geq \ell_{a,c}^\theta + \ell_{c,b}^\theta \geq \ell_{a,b}^\theta. \quad \square$$

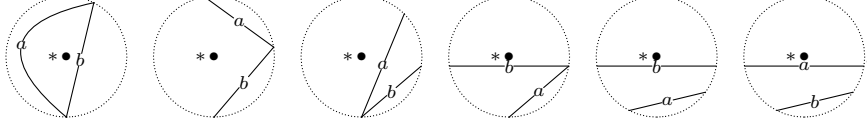
In the end of this section, we will prove that Γ_σ has the structure of an R -order and we will specify its structure. More precisely, we prove that the center of Γ_σ is isomorphic to R' and we realize Γ_σ as a matrix algebra whose entries are free R -submodules of \mathcal{R}' (see (2.16), page 9).

For each vertex d of Q'_σ , we will define an element C_d of $e_d \Gamma_\sigma e_d$ as follows.

- If Q'_σ does not contain Figure 2.10 as a subtriangulation, all the planar cycles at d are equivalent (because of commutativity relations). Denote by C_d the common value of these planar cycles in Γ_σ .

- If Q'_σ contains Figure 2.10 as a subtriangulation, if $d \neq j$, all cycles at d that are planar in the full subquiver $Q'_\sigma \setminus \{j\}$ are equivalent. Denote by C_d their common value in Γ_σ . We define $C_j = 0$.

For each frozen vertex a of Q'_σ , we denote by E_a the big cycle at a passing through each external arrow once. Finally, if a and b are two vertices of Q'_σ corresponding to edges which are not incident to the puncture, we denote $a \vdash b$ if $a_2 \in \llbracket b_2, a_1 \rrbracket$ or $b_1 \in \llbracket b_2, a_1 \rrbracket$. We have $a \vdash b$ in the following cases:



Let us fix a grading of \mathcal{R}' such that u and v have degree 1. Then, R' is graded as a subalgebra of \mathcal{R}' (X and Y have degree $2n$). If a and b are two vertices of Q'_σ , we consider a graded R' -submodule $A_{a,b}$ of \mathcal{R}' (that is also free over R) defined in the following way:

- $A_{a,b} = 0$ if a and b are incident to the puncture and tagged differently (as i and j in Figure 2.10);
- $A_{a,b} = u^{\ell_{a,b}^\theta} v R'$ if one of a and b is incident to the puncture and plain and the other one either incident to the puncture and plain either not incident to the puncture;
- $A_{a,b} = u^{\ell_{a,b}^\theta} (u - v) R'$ if one of a and b is incident to the puncture and notched and the other one either incident to the puncture and notched either not incident to the puncture;
- $A_{a,b} = u^{\ell_{a,b}^\theta} R' + v^{\ell_{a,b}^\theta} R'$ if a and b are not incident to the puncture and $a \vdash b$;
- $A_{a,b} = u^{\ell_{a,b}^\theta} R'$ if a and b are not incident to the puncture and $a \not\vdash b$.

It is an easy consequence of Lemma 2.15 that $A := (A_{a,b})_{a,b \in Q'_{\sigma,0}}$ is an R -subalgebra of the matrix algebra $M_{Q'_{\sigma,0}}(\mathcal{R}')$.

Proposition 2.26. *There exists an isomorphism of graded algebras $\phi_\sigma : R' \rightarrow Z(\Gamma_\sigma)$ ($Z(\Gamma_\sigma)$ is graded by θ -length). Moreover, for the induced R' -algebra structure of Γ_σ , there is an isomorphism of graded R' -algebras $\psi_\sigma : A \rightarrow \Gamma_\sigma$ induced by isomorphisms of graded R' -modules*

$$\psi_\sigma^{a,b} : A_{a,b} \xrightarrow{\sim} e_a \Gamma_\sigma e_b$$

(Γ_σ is graded by θ -length).

Finally, the following properties are satisfied:

- (i) $_\sigma$ for each frozen vertex a of Q_σ ,

$$e_a \phi_\sigma(X) = \phi_\sigma(X) e_a = E_a$$

- (ii) $_\sigma$ for each vertex a of Q_σ ,

$$\begin{aligned} e_a \phi_\sigma(Y) &= \phi_\sigma(Y) e_a \\ &= \begin{cases} e_a \phi_\sigma(X) - C_a & \text{if } \sigma \text{ has no plain arc incident to the puncture,} \\ C_a & \text{else;} \end{cases} \end{aligned}$$

- (iii) $_\sigma$ for any pair of frozen vertices a and b , $\psi_\sigma^{a,b}(u^{\ell_{a,b}^\theta})$ is equivalent to the shortest path among paths consisting of external arrows from a to b ;

- (iv) $_\sigma$ for any pair of frozen vertices a and b such that a follows immediately b in the anticlockwise order ($b_2 = a_1$), let $s_{a,b}$ be the path from a to b whose

composition with the external arrow $b \rightarrow a$ is the anticlockwise external cycle winding around a_1 ; then we have

$$\psi_\sigma^{a,b}(v^{\ell_{a,b}^\theta}) = \begin{cases} \psi_\sigma^{a,b}(u^{\ell_{a,b}^\theta}) - s_{a,b} & \text{if } \sigma \text{ has no plain arc incident to the puncture} \\ s_{a,b} & \text{else;} \end{cases}$$

(v) $_\sigma$ for any external arrow α of Q'_σ , and any $w \in \Gamma_\sigma$, if $\alpha w = 0$ then $e_{t(\alpha)}w = 0$ and if $w\alpha = 0$ then $w e_{s(\alpha)} = 0$.

First of all, when the triangulation has only notched arcs incident to the puncture, the situation is similar to the fully plain one. Then, up to applying the R -automorphism of R' given by $Y \mapsto X - Y$ and the $K[u^{\pm 1}]$ -automorphism of \mathcal{R}' given by $v \mapsto u - v$, both results are equivalent (note that this pair of automorphisms commutes with the inclusion $R' \subset \mathcal{R}'$). From now on, we will only look at cases where triangulations have at most one notched arc incident to the puncture.

Remark that (v) $_\sigma$ is in fact implied by the rest of the proposition. Indeed, suppose that α is an external arrow from a vertex a to a vertex b and suppose that $w \in \Gamma_\sigma$ satisfies $\alpha w = 0$. Without loss of generality, we can suppose that $w = e_b w e_c$ for some vertex c of Q'_σ . Thus, there is an element $p \in A_{b,c}$ such that $w = \psi_\sigma^{b,c}(p)$. Then, thanks to (i) $_\sigma$ and the multiplicativity of ψ_σ , we have

$$0 = E_b w = \phi_\sigma(X) e_b \psi_\sigma^{b,c}(p) = \phi_\sigma(X) \psi_\sigma^{b,c}(p) = \psi_\sigma^{b,c}(Xp)$$

and, as $\psi_\sigma^{b,c}$ is injective, we get $Xp = 0$ in $A_{b,c}$. Since $A_{b,c}$ is free over $R \subset R'$, it follows that $p = 0$ and therefore $w = 0$. The other case is dealt in the same way.

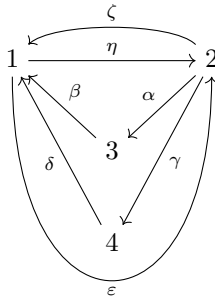
If we supposed that the existence of ϕ_σ as a morphism of algebra and (ii) $_\sigma$ are proved, we can easily deduced that ϕ_σ is graded. Indeed, we proved in Proposition 2.24 that $\ell^\theta(C_k) = 2n$. Thus, thanks to (ii) $_\sigma$, if σ has at most one notched edge incident to the puncture, $\phi_\sigma(Y)$ is homogeneous of degree $2n$. Moreover, as ϕ_σ is a morphism of algebra, we have $\phi_\sigma(X)\phi_\sigma(Y) = \phi_\sigma(Y)\phi_\sigma(X)$ so $\phi_\sigma(X)$ is also homogeneous of degree $2n$.

It is then automatic that $\psi_\sigma^{a,b}$ is graded for any vertices a and b (under the hypothesis that $\psi_\sigma^{a,b}$ is an isomorphism of R' -modules). Indeed, $\ell_{a,b}^\theta$ is by definition the minimal θ -length of a path from a to b .

The strategy for the rest of the proposition is to do an induction on n . We start by proving the proposition for two families of initial cases.

Lemma 2.27. *Suppose that $n = 2$ and the triangulation σ consists of one plain and one notched arc incident to the puncture as in Figure 2.10. Then Proposition 2.26 is satisfied.*

Proof. In this case, the quiver Q'_σ is



and the potential is $W'_\sigma = \eta\alpha\beta + \eta\gamma\delta - \gamma\delta\varepsilon - \eta\zeta$ so the relations are $\beta\eta = \eta\alpha = 0$, $\delta\eta = \delta\varepsilon$, $\eta\gamma = \varepsilon\gamma$ and $\zeta = \alpha\beta + \gamma\delta$ (3 corresponds to the notched arc and 4 to the plain one). Using these relations we get

Claim 1. Any path different from η can be, up to the relations, expressed in a unique way without subpaths of the form η , ζ , $\beta\varepsilon\gamma$ or $\delta\varepsilon\alpha$ (the last two are 0 in Γ_σ).

The θ -lengths are given by $\ell^\theta(\alpha) = \ell^\theta(\beta) = \ell^\theta(\gamma) = \ell^\theta(\delta) = 1$ and $\ell^\theta(\varepsilon) = \ell^\theta(\zeta) = \ell^\theta(\eta) = 2$.

Let us prove that there is an isomorphism

$$\begin{aligned}\phi_\sigma : R' &\rightarrow Z(\Gamma_\sigma) \\ X &\mapsto \varepsilon\zeta + \zeta\varepsilon + \beta\varepsilon\alpha + \delta\varepsilon\gamma \\ Y &\mapsto \varepsilon\gamma\delta + \gamma\delta\varepsilon + \delta\varepsilon\gamma.\end{aligned}$$

It is easy to see that the two elements $\phi_\sigma(X)$ and $\phi_\sigma(Y)$ commute with all arrows so the image is included in the center. Moreover, we have

$$\begin{aligned}\phi_\sigma(X)\phi_\sigma(Y) &= \varepsilon\alpha\beta\varepsilon\gamma\delta + \varepsilon\gamma\delta\varepsilon\gamma\delta + \alpha\beta\varepsilon\gamma\delta\varepsilon + \gamma\delta\varepsilon\gamma\delta\varepsilon + \delta\varepsilon\gamma\delta\varepsilon\gamma \\ &= \varepsilon\gamma\delta\varepsilon\gamma\delta + \gamma\delta\varepsilon\gamma\delta\varepsilon + \delta\varepsilon\gamma\delta\varepsilon\gamma = \phi_\sigma(Y)^2\end{aligned}$$

so ϕ_σ is a morphism. Notice that

$$\begin{aligned}\phi_1 = e_1\phi_\sigma e_1 : R' &\rightarrow e_1Z(\Gamma_\sigma)e_1 \\ X &\mapsto \varepsilon\alpha\beta + \varepsilon\gamma\delta \\ Y &\mapsto \varepsilon\gamma\delta\end{aligned}$$

is an isomorphism. Indeed, the surjectivity comes from Claim 1. For the injectivity, notice that every element of R' can be written (uniquely) in the form $P + YQ$ where P and Q are polynomials in the variable X . Then, as $\varepsilon\alpha\beta\varepsilon\gamma\delta = \varepsilon\gamma\delta\varepsilon\alpha\beta = 0$,

$$\phi_1(P + YQ) = P(\varepsilon\alpha\beta) + P(\varepsilon\gamma\delta) - P(0)e_1 + \varepsilon\gamma\delta Q(\varepsilon\gamma\delta).$$

If $\phi_1(P + YQ) = 0$ then $\beta\phi_1(P + YQ) = \beta P(\varepsilon\alpha\beta) = 0$. As $\Gamma_\sigma/(e_4, \eta)$ is a path algebra (all relations are in the ideal (e_4, η) of KQ'_σ except $\zeta = \alpha\beta + \gamma\delta$), we get $P = 0$. Then $\varepsilon\gamma\delta Q(\varepsilon\gamma\delta) = 0$. As $\Gamma_\sigma/(e_3, \eta - \varepsilon)$ is a path algebra (all relations are in the ideal $(e_3, \eta - \varepsilon)$ of KQ'_σ except $\zeta = \alpha\beta + \gamma\delta$), we get $Q = 0$. Thus ϕ_1 is injective. We deduce immediately that ϕ_σ is also injective.

For the surjectivity of ϕ_σ , take an element z of $Z(\Gamma_\sigma)$. Using Claim 1, it is immediate that we can write

$$z = P_1(\varepsilon\alpha\beta) + Q_1(\varepsilon\gamma\delta) + P_2(\alpha\beta\varepsilon) + Q_2(\gamma\delta\varepsilon) + P_3(\beta\varepsilon\alpha) + Q_4(\delta\varepsilon\gamma)$$

where Q_1 and Q_2 have no constant terms (we take the convention that $(\varepsilon\alpha\beta)^0 = e_1$, $(\alpha\beta\varepsilon)^0 = e_2$, $(\beta\varepsilon\alpha)^0 = e_3$ and $(\delta\varepsilon\gamma)^0 = e_4$). Using the identity $\alpha z = z\alpha$, as $\delta\varepsilon\alpha = 0$, we get $P_2(\alpha\beta\varepsilon)\alpha = \alpha P_3(\beta\varepsilon\alpha)$ and, thanks to the grading by ℓ^θ , $P_2 = P_3$. In the same way, $\beta z = z\beta$ implies $P_1 = P_3$, $\gamma z = z\gamma$ implies $Q_2 = Q_4$ and $\delta z = z\delta$ implies $Q_1 = Q_4$. So $z = P_1(\varepsilon\alpha\beta + \alpha\beta\varepsilon + \beta\varepsilon\alpha) + Q_1(\varepsilon\gamma\delta + \gamma\delta\varepsilon + \delta\varepsilon\gamma) = \phi_\sigma(P_1(X - Y) + Q_1(Y))$.

It is an easy observation, using Claim 1, that $e_1Z(\Gamma_\sigma)e_i = e_i\Gamma_\sigma e_i$ for every $i \in Q'_{\sigma,0}$. This permits to compute, together with θ -lengths given at the beginning, the following isomorphisms of R' -modules (denoted $\psi_\sigma^{a,b}$) from $A_{a,b}$ to $e_a\Gamma_\sigma e_b$ where $a, b \in \llbracket 1, 4 \rrbracket$:

$a \backslash b$	1	2	3	4
1	$1 \mapsto e_1$	$u^2 \mapsto \varepsilon, v^2 \mapsto \eta$	$(u - v)^3 \mapsto \varepsilon\alpha$	$v^3 \mapsto \varepsilon\gamma$
2	$u^2 \mapsto \zeta, v^2 \mapsto \gamma\delta$	$1 \mapsto e_2$	$u - v \mapsto \alpha$	$v \mapsto \gamma$
3	$u - v \mapsto \beta$	$(u - v)^3 \mapsto \beta\varepsilon$	$u^{-1}(u - v) \mapsto e_3$	0
4	$v \mapsto \delta$	$v^3 \mapsto \delta\varepsilon$	0	$u^{-1}v \mapsto e_4$

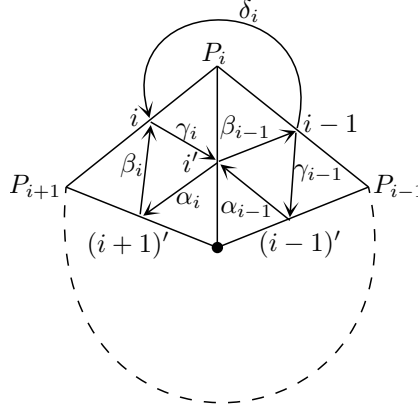


FIGURE 2.29. Initial case

(note that $1 \vdash 2$ and $2 \vdash 1$). The points $(i)_\sigma$ to $(iv)_\sigma$ are easy to check. The multiplicativity can be checked case by case easily. \square

Lemma 2.28. *Suppose that the triangulation consists of the (plain) arcs connecting each vertex of the polygon with the puncture. Then Proposition 2.26 is satisfied.*

Proof. For each i from 1 to n , let us denote by i the arc from P_i to P_{i+1} and by i' the arc from P_i to the puncture. Let us call α_i the arrow of Q_σ from i' to $(i+1)'$, β_i the arrow from $(i+1)'$ to i , γ_i the arrow from i to i' and δ_i the arrow from $i-1$ to i (modulo n) (Figure 2.29). Remark that $\ell^\theta(\alpha_i) = \ell^\theta(\delta_i) = 2$, $\ell^\theta(\beta_i) = \ell^\theta(\gamma_i) = n-1$. The relations in Γ_σ are $\beta_i\gamma_i = \alpha_{i+1}\alpha_{i+2}\cdots\alpha_{i-2}\alpha_i$, $\gamma_i\alpha_i = \delta_{i+1}\gamma_{i+1}$ and $\alpha_i\beta_i = \beta_{i-1}\delta_i$ for all $i \in \llbracket 1, n \rrbracket$.

Notice that any path is equivalent to a path containing only arrows of type δ or to a path containing no arrow of type δ . Then, up to equivalence, a path containing no arrow of type δ can be supposed not to contain any arrow of type γ , except maybe at the beginning and not to contain any arrow of type β except maybe at the end. To summarize:

Claim 2. Any path of Q'_σ is equivalent to a path of the form

$$\delta_i\delta_{i+1}\cdots\delta_j \quad \text{or} \quad \gamma_i^\mu\alpha_i\alpha_{i+1}\cdots\alpha_j\beta_j^\nu,$$

where $\mu, \nu \in \{0, 1\}$.

For $i \in \llbracket 1, n \rrbracket$, we denote $E_i = \delta_{i+1}\delta_{i+2}\cdots\delta_i$, $C_i = \gamma_i\beta_{i-1}\delta_i = \gamma_i\alpha_i\beta_i = \delta_{i+1}\gamma_{i+1}\beta_i$ and $C_{i'} = \beta_{i-1}\delta_i\gamma_i = \beta_{i-1}\gamma_{i-1}\alpha_{i-1} = \alpha_i\alpha_{i+1}\cdots\alpha_{i-1} = \alpha_i\beta_i\gamma_i$. Finally, we denote

$$E = \sum_{i=1}^n (E_i + C_{i'}) \quad \text{and} \quad C = \sum_{i=1}^n (C_i + C_{i'}).$$

Let us prove that there is an isomorphism of algebras given by

$$\begin{aligned} \phi_\sigma : R' = K[X, Y]/(YX - Y^2) &\rightarrow Z(\Gamma_\sigma) \\ X &\mapsto E \\ Y &\mapsto C. \end{aligned}$$

We get easily from the relations that $C_{i'}\alpha_i = \alpha_i C_{(i+1)'}$, $C_{(i+1)'}\beta_i = \beta_i C_i = \beta_i E_i$, $C_i\gamma_i = E_i\gamma_i = \gamma_i C_{i'}$, $\delta_i E_i = E_{i-1}\delta_i$, $\delta_i C_i = C_{i-1}\delta_i$, $\beta_i C_i = \beta_i E_i$, $C_i\gamma_i = E_i\gamma_i$ and

$C_i E_i = C_i^2$. Therefore C and E are in the center of Γ_σ and ϕ_σ is a morphism of algebras.

Any element of R' can be written as $P(X) + YQ(Y)$ where P and Q are two polynomials. If this element is in $\ker \phi_\sigma$, $P(E) + CQ(C) = 0$. Notice now that a path which contains only arrows of type δ is not related to any other path by the relations. Thus, we should have $P(E) = 0$ and then $P = 0$ as a polynomial (paths appearing in $CQ(C)$ contain arrows which are not of type δ). Then $CQ(C) = 0$. Powers of C have different θ -lengths so $Q = 0$ and finally $\ker \phi_\sigma = 0$.

Let us prove that ϕ_σ is surjective. Let $z \in Z(\Gamma_\sigma)$. Using Claim 2 and $E_i C_i = C_i^2$, we can write

$$z = \sum_{i=1}^n [P_i(E_i) + Q_i(C_i) + S_i(C_{i'})]$$

for some polynomials P_i , Q_i and S_i where Q_i has no constant term (we take the convention that $E_i^0 = e_i$ and $C_{i'}^0 = e_{i'}$). For any i , $z\alpha_i = \alpha_i z$ implies that $S_i(C_{i'})\alpha_i = \alpha_i S_{i+1}(C_{(i+1)'})$, so using the grading by θ -length, $S_i = S_{i+1}$. In the same way, $z\beta_i = \beta_i z$ implies $S_{i+1}(C_{(i+1)'})\beta_i = \beta_i (P_i(E_i) + Q_i(C_i)) = (P_i(C_{(i+1)'}) + Q_i(C_{(i+1)'})\beta_i$ because $\beta_i E_i = \beta_i C_i = C_{(i+1)'}\beta_i$ so we get $S_{i+1} = P_i + Q_i$. Finally, $z\delta_i = \delta_i z$ implies $(P_{i-1}(E_{i-1}) + Q_{i-1}(C_{i-1}))\delta_i = \delta_i (P_i(E_i) + Q_i(C_i))$. As already observed before, there are no relations between paths containing only δ 's and other paths. Thus, $P_{i-1}(E_{i-1})\delta_i = \delta_i P_i(E_i)$ and $Q_{i-1}(C_{i-1})\delta_i = \delta_i Q_i(C_i)$ and using θ -length, $P_{i-1} = P_i$ and $Q_{i-1} = Q_i$. Finally, we get

$$z = P_1 \left(\sum_{i=1}^n E_i \right) + Q_1 \left(\sum_{i=1}^n C_i \right) + (P_1 + Q_1) \left(\sum_{i=1}^n C_{i'} \right) = \phi_\sigma(P_1(X) + Q_1(Y))$$

so ϕ_σ is surjective.

Let now i, j be two frozen vertices. Notice that

$$i \vdash j \Leftrightarrow i+1 \in \llbracket j+1, i \rrbracket \text{ or } j \in \llbracket j+1, i \rrbracket \Leftrightarrow j = i-1.$$

The following maps are isomorphisms of graded R' -modules:

$$\begin{aligned} \psi_\sigma^{i,j} : u^{2d(i,j)} R' &\rightarrow e_i \Gamma_\sigma e_j & (j \neq i-1) \\ u^{2d(i,j)} &\mapsto \delta_{i+1} \delta_{i+2} \dots \delta_j; \end{aligned}$$

$$\begin{aligned} \psi_\sigma^{i,i-1} : u^{2(n-1)} R' + v^{2(n-1)} R' &\rightarrow e_i \Gamma_\sigma e_{i-1} \\ u^{2(n-1)} &\mapsto \delta_{i+1} \delta_{i+2} \dots \delta_{i-1} \\ v^{2(n-1)} &\mapsto \gamma_i \beta_{i-1}; \end{aligned}$$

$$\begin{aligned} \psi_\sigma^{i,j'} : v^{2d(i,j)+n-1} R' &\rightarrow e_i \Gamma_\sigma e_{j'} \\ v^{2d(i,j)+n-1} &\mapsto \gamma_i \alpha_i \dots \alpha_{j-1}; \end{aligned}$$

$$\begin{aligned} \psi_\sigma^{i',j} : v^{2d(i,j+1)+n-1} R' &\rightarrow e_{i'} \Gamma_\sigma e_j \\ v^{2d(i,j+1)+n-1} &\mapsto \alpha_i \dots \alpha_j \beta_j; \end{aligned}$$

$$\begin{aligned} \psi_\sigma^{i',j'} : v^{2d(i,j)} R' &\rightarrow e_{i'} \Gamma_\sigma e_{j'} & (j \neq i) \\ v^{2d(i,j)} &\mapsto \alpha_i \dots \alpha_{j-1}; \end{aligned}$$

$$\begin{aligned}\psi_{\sigma}^{i',i'} : u^{-1}vR' &\rightarrow e_{i'}\Gamma_{\sigma}e_{i'} \\ u^{-1}v &\mapsto e_{i'}.\end{aligned}$$

The argument mainly relies on Claim 2. Let us for example look at the second case. It is easy to check that

$$\psi_{\sigma}^{i,i-1} \left(v^{2(n-1)} \right) (E - C) = 0 \quad \text{and} \quad \psi_{\sigma}^{i,i-1} \left(u^{2(n-1)} - v^{2(n-1)} \right) C = 0,$$

so $\psi_{\sigma}^{i,i-1}$ is a morphism. Moreover, if an element $u^{2(n-1)}P(X) + v^{2(n-1)}Q(Y)$ is mapped to 0 by this map, using the same kind of analysis than before, we prove that $P = Q = 0$ so the map is injective. For the surjectivity, notice that, for any path that does not contain δ from i to $i-1$, different from $\gamma_i\beta_{i-1}$, in the form given by Claim 2, we can write

$$\begin{aligned}\gamma_i\alpha_i \dots \alpha_{i-1}\beta_{i-1} &= \delta_{i+1}\gamma_{i+1}\alpha_{i+1} \dots \alpha_{i-1}\beta_{i-1} = \dots \\ &= \delta_{i+1} \dots \delta_{i-1}\gamma_{i-1}C_{i-1}^k\alpha_{i-1}\beta_{i-1} = \delta_{i+1} \dots \delta_{i-1}C_{i-1}^{k+1}\end{aligned}$$

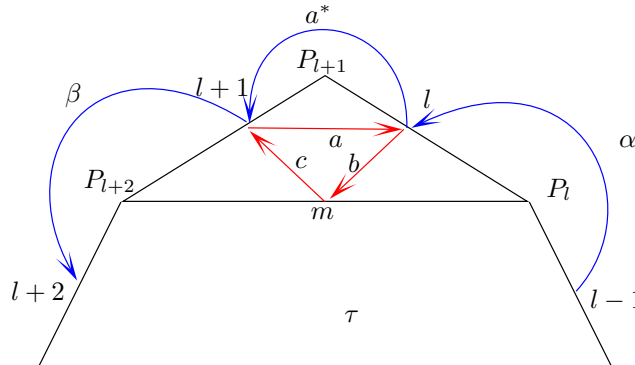
so the map is surjective.

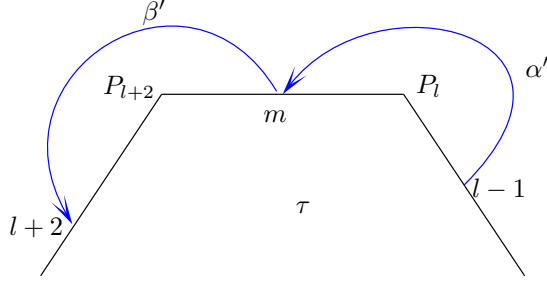
The multiplicativity can be checked case by case. For example, if $i, j, k \in \llbracket 1, n \rrbracket$ and $j \in \llbracket i, k \rrbracket$, we have

$$\begin{aligned}\psi_{\sigma}^{i,j'}(v^{2d(i,j)+n-1})\psi_{\sigma}^{j',k}(v^{2d(j,k+1)+n-1}) &= \gamma_i\alpha_i\alpha_{i+1} \dots \alpha_{j-1}\alpha_j \dots \alpha_k\beta_k \\ &= \delta_{i+1}\gamma_{i+1}\alpha_{i+1} \dots \alpha_k\beta_k \\ &= \dots = \delta_{i+1} \dots \delta_k\gamma_k\alpha_k\beta_k \\ &= \delta_{i+1} \dots \delta_k C = \psi_{\sigma}^{i,k}(u^{2d(i,k)}Y) \\ &= \psi_{\sigma}^{i,k}(v^{2d(i,k)+2n}) \\ &= \psi_{\sigma}^{i,k}(v^{2d(i,j)+n-1}v^{2d(j,k+1)+n-1})\end{aligned}$$

The points $(i)_{\sigma}$ to $(iv)_{\sigma}$ are easy to check (and recall that $(v)_{\sigma}$ is a consequence of them). \square

Proof of Proposition 2.26. Let us suppose that the result is proved for all triangulations of polygons with $n-1$ vertices and that there is a *corner triangle* $P_l P_{l+1} P_{l+2}$ in the triangulation as follows:





(if there is no corner triangle, we are either in the case of Lemma 2.27 either in the case of Lemma 2.28). By induction hypothesis, the expected results hold for τ . As the θ -length depends on the size of the polygon, we will denote by $\ell^{\theta, \tau}$ (resp. ℓ^θ) the θ -length in τ (resp. σ). In the same way, as the inclusion $R' \subset \mathcal{R}'$ depends on the triangulation, we will call u_τ and v_τ the generators of \mathcal{R}' when we consider this inclusion for τ .

Then there is a non-unital monomorphism

$$\begin{aligned} \xi : KQ'_\tau &\hookrightarrow KQ'_\sigma \\ \alpha' &\mapsto \alpha b, \\ \beta' &\mapsto c\beta, \\ \gamma &\mapsto \gamma \text{ for } \gamma \in (Q'_\tau)_1 \setminus \{\alpha', \beta'\}. \end{aligned}$$

We have $W'_\sigma = \xi(W'_\tau) + abc - aa^*$, so for any $\gamma \in Q'_\tau$ which is not external, $\partial_\gamma W'_\sigma = \xi(\partial_\gamma W'_\tau)$. Thus ξ induces a morphism $\bar{\xi} : \Gamma_\tau \rightarrow \Gamma_\sigma$. Notice that using the relations, any path of Q'_σ is equivalent to an element which does not contain a^* , ca or ab . It is then easy to see that:

Claim 3. Any path of Q'_σ is equivalent to a path of one of the following forms where ω is a path of Q'_τ : (1) $\xi(\omega)$; (2) $\xi(\omega)\alpha$; (3) $\xi(\omega)c$; (4) $b\xi(\omega)$; (5) $\beta\xi(\omega)$; (6) $b\xi(\omega)\alpha$; (7) $b\xi(\omega)c$; (8) a ; (9) $\beta\xi(\omega)\alpha$; (10) $\beta\xi(\omega)c$.

Let us prove that $\bar{\xi}$ is in fact injective. Let π_τ be the canonical projection $KQ'_\tau \rightarrow \Gamma_\tau$. Thanks to $(iv)_\tau$, we get:

$$\begin{aligned} \ker \bar{\xi} &= \pi_\tau(\xi^{-1}(\langle \partial_b W'_\sigma, \partial_c W'_\sigma \rangle + \xi(\mathcal{J}'(W'_\tau)))) \\ &= \pi_\tau(\xi^{-1}(\langle \partial_b W'_\sigma, \partial_c W'_\sigma \rangle) + \xi^{-1}(\xi(\mathcal{J}'(W'_\tau)))) \\ &= \pi_\tau(\xi^{-1}(\langle \partial_b W'_\sigma, \partial_c W'_\sigma \rangle) + \pi_\tau(\mathcal{J}'(W'_\tau))) \\ &= \pi_\tau(\xi^{-1}(\langle ca - \omega_{m, l-1}\alpha, ab - \beta\omega_{l+2, m} \rangle)) \end{aligned}$$

where $\omega_{m, l-1} = \psi_\tau^{m, l-1} \left(v_\tau^{\ell^{\theta, \tau}_{m, l-1}} \right)$ and $\omega_{l+2, m} = \psi_\tau^{l+2, m} \left(v_\tau^{\ell^{\theta, \tau}_{l+2, m}} \right)$.

Up to equivalence, a path of Q'_σ can always be supposed not to contain a^* . Moreover, $e_m \text{Im}(\xi)e_l = 0 = e_{l+1} \text{Im}(\xi)e_m$, so

$$\xi^{-1}(\langle ca - \omega_{m, l-1}\alpha, ab - \beta\omega_{l+2, m} \rangle) = \xi^{-1}(\langle cab - \omega_{m, l-1}\alpha b, cab - c\beta\omega_{l+2, m} \rangle)$$

$$\begin{aligned} \text{and } \ker \bar{\xi} &= \pi_\tau(\xi^{-1}(\langle cab - \omega_{m, l-1}\alpha b, cab - c\beta\omega_{l+2, m} \rangle)) \\ &= \pi_\tau(\xi^{-1}(\langle c\beta\omega_{l+2, m} - \omega_{m, l-1}\alpha b, cab - c\beta\omega_{l+2, m} \rangle)) \\ &= \pi_\tau(\langle \beta'\omega_{l+2, m} - \omega_{m, l-1}\alpha', \xi^{-1}(\langle cab - c\beta\omega_{l+2, m} \rangle) \rangle) \\ &= \pi_\tau(\xi^{-1}(\langle cab - c\beta\omega_{l+2, m} \rangle)) = \pi_\tau(0) = 0. \end{aligned}$$

Let us prove that the following map is an isomorphism:

$$\begin{aligned}\phi_\sigma : Z(\Gamma_\tau) &\cong R' \rightarrow Z(\Gamma_\sigma) \\ X &\mapsto \bar{\xi}(\phi_\tau(X)) + E_l + E_{l+1} \\ Y &\mapsto \bar{\xi}(\phi_\tau(Y)) + C_l + C_{l+1}\end{aligned}$$

First of all, $\phi_\sigma(X), \phi_\sigma(Y) \in Z(\Gamma_\sigma)$. Indeed, for each arrow γ in Q_τ , we have, by using induction hypothesis,

$$\gamma\phi_\sigma(X) = \gamma\bar{\xi}(\phi_\tau(X)) = \bar{\xi}(\gamma\phi_\tau(X)) = \bar{\xi}(\phi_\tau(X)\gamma) = \bar{\xi}(\phi_\tau(X))\gamma = \phi_\sigma(X)\gamma$$

and the same for Y . For arrows which are not in Q_τ we have

$$\begin{aligned}a\phi_\sigma(X) &= aE_l = aa^*\beta \overset{\text{ext.}}{\curvearrowright} \alpha = C_{l+1}\beta \overset{\text{ext.}}{\curvearrowright} \alpha \\ &= \beta C_{l+2} \overset{\text{ext.}}{\curvearrowright} \alpha = \cdots = \beta \overset{\text{ext.}}{\curvearrowright} C_l = E_{l+1}a = \phi_\sigma(X)a\end{aligned}$$

where ext. denote products of external arrows and, thanks to $(i)_\tau$,

$$\begin{aligned}b\phi_\sigma(X) &= bc\beta \overset{\text{ext.}}{\curvearrowright} \alpha b = E_lb = \phi_\sigma(X)b; \\ c\phi_\sigma(X) &= cE_{l+1} = c\beta \overset{\text{ext.}}{\curvearrowright} \alpha bc = \phi_\sigma(X)c; \\ \beta\phi_\sigma(X) &= \beta E_{l+2} = E_{l+1}\beta = \phi_\sigma(X)\beta; \\ \alpha\phi_\sigma(X) &= \alpha E_l = E_{l-1}\alpha = \phi_\sigma(X)\alpha.\end{aligned}$$

For $\phi_\sigma(Y)$, using $(ii)_\tau$, all the computations are immediate.

Notice now that

$$\begin{aligned}C_l^2 &= bc(ab)(ca) = bc\beta\bar{\xi}\left(\psi_\tau^{l+2,m}\left(v_\tau^{\ell_{l+2,m}^{\theta,\tau}}\right)\right)\bar{\xi}\left(\psi_\tau^{m,l-1}\left(v_\tau^{\ell_{m,l-1}^{\theta,\tau}}\right)\right)\alpha \\ &= bc\beta\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(v_\tau^{\ell_{l+2,l-1}^{\theta,\tau}+2n}\right)\right)\alpha = bc\beta\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(Yu_\tau^{\ell_{l+2,l-1}^{\theta,\tau}}\right)\right)\alpha \\ &= bc\beta C_{l+2}\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(u_\tau^{\ell_{l+2,l-1}^{\theta,\tau}}\right)\right)\alpha \\ &= C_l bc\beta\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(u_\tau^{\ell_{l+2,l-1}^{\theta,\tau}}\right)\right)\alpha = C_l E_l\end{aligned}$$

and $C_{l+1}^2 = C_{l+1}E_{l+1}$ by the same method. Therefore

$$\phi_\sigma(YX) = \bar{\xi}(\phi_\tau(Y))\bar{\xi}(\phi_\tau(X)) + C_l E_l + C_{l+1} E_{l+1} = \bar{\xi}(\phi_\tau(Y^2)) + C_l^2 + C_{l+1}^2 = \phi_\sigma(Y^2)$$

so ϕ_σ is a morphism. As $\bar{\xi}$ and ϕ_τ are injective, ϕ_σ is also injective. The last thing to show is that ϕ_σ is surjective.

Using Claim 3 and that, as it commutes with idempotents, every element $z \in Z(\Gamma_\sigma)$ is a linear combination of cycles, we can write

$$z = \bar{\xi}(z') + b\bar{\xi}(z'')\alpha + \beta\bar{\xi}(z''')c.$$

Then, as z is in the center, for any $x \in \Gamma_\tau$,

$$\bar{\xi}(xz') = \bar{\xi}(x)\bar{\xi}(z') = \bar{\xi}(x)z = z\bar{\xi}(x) = \bar{\xi}(z')\bar{\xi}(x) = \bar{\xi}(z'x)$$

and, as $\bar{\xi}$ is injective, $xz' = z'x$ and $z' \in Z(\Gamma_\tau)$. Therefore, up to subtracting $\phi_\sigma(z')$, we can suppose that $z = b\bar{\xi}(z'')\alpha + \beta\bar{\xi}(z''')c$. Hence, we have

$$0 = z\alpha b = \alpha z b = \bar{\xi}(\alpha' z'' \alpha')$$

and, as $\bar{\xi}$ is injective, $\alpha' z'' \alpha' = 0$. Finally, $e_m z'' e_{l-1} = 0$ thanks to $(v)_\tau$. In the same way $e_{l+2} z''' e_m = 0$ so $z = 0$. Therefore ϕ_σ is surjective.

We will prove the existence of ψ_σ as a family of morphisms of R' -modules. Then, these morphisms are automatically graded by looking at homogeneous generators. If $i, j \notin \{l, l+1\}$, $\bar{\xi}$ induces an isomorphism of R' -modules from $e_i \Gamma_\tau e_j$ to $e_i \Gamma_\sigma e_j$. It proves the existence of $\psi_\sigma^{i,j}$ in this case.

Let us recall that $l = (P_l, P_{l+1})$, $l+1 = (P_{l+1}, P_{l+2})$ and $m = (P_l, P_{l+2})$. Thus, $l+1 \vdash l$, $l \nvdash l+1$, and for any $i \in Q'_{\sigma,0} \setminus \{l, l+1\}$ which is not incident to the puncture, we have $i_1, i_2 \neq l+1$, so

$$\begin{aligned} i \vdash l &\Leftrightarrow i_2 \in \llbracket l+1, i_1 \rrbracket \text{ or } l \in \llbracket l+1, i_1 \rrbracket \\ &\Leftrightarrow i_2 \in \llbracket l, i_1 \rrbracket \text{ or } l-1 \in \llbracket l, i_1 \rrbracket \Leftrightarrow i \vdash l-1 \end{aligned}$$

$$\begin{aligned} l \vdash i &\Leftrightarrow l+1 \in \llbracket i_2, l \rrbracket \text{ or } i_1 \in \llbracket i_2, l \rrbracket \\ &\Leftrightarrow l+2 \in \llbracket i_2, l \rrbracket \text{ or } i_1 \in \llbracket i_2, l \rrbracket \Leftrightarrow m \vdash i \end{aligned}$$

$$\begin{aligned} i \vdash l+1 &\Leftrightarrow i_2 \in \llbracket l+2, i_1 \rrbracket \text{ or } l+1 \in \llbracket l+2, i_1 \rrbracket \\ &\Leftrightarrow i_2 \in \llbracket l+2, i_1 \rrbracket \text{ or } l \in \llbracket l+2, i_1 \rrbracket \Leftrightarrow i \vdash m \end{aligned}$$

$$\begin{aligned} l+1 \vdash i &\Leftrightarrow l+2 \in \llbracket i_2, l+1 \rrbracket \text{ or } i_1 \in \llbracket i_2, l+1 \rrbracket \\ &\Leftrightarrow l+3 \in \llbracket i_2, l+2 \rrbracket \text{ or } i_1 \in \llbracket i_2, l+2 \rrbracket \Leftrightarrow l+2 \vdash i. \end{aligned}$$

Let $i \notin \{l, l+1\}$. There are isomorphisms of R' -modules

$$\begin{aligned} e_i \Gamma_\tau e_{l-1} &\rightarrow e_i \Gamma_\sigma e_l & e_{l+2} \Gamma_\tau e_i &\rightarrow e_{l+1} \Gamma_\sigma e_i \\ \omega &\mapsto \bar{\xi}(\omega)\alpha & \omega &\mapsto \beta \bar{\xi}(\omega) \\ e_m \Gamma_\tau e_i &\rightarrow e_l \Gamma_\sigma e_i & e_i \Gamma_\tau e_m &\rightarrow e_i \Gamma_\sigma e_{l+1} \\ \omega &\mapsto b \bar{\xi}(\omega) & \omega &\mapsto \bar{\xi}(\omega)c \end{aligned}$$

The injectivity comes from $(v)_\tau$. For example, if $\bar{\xi}(\omega)\alpha = 0$ then $\bar{\xi}(\omega)\alpha b = 0$ and therefore $\bar{\xi}(\omega\alpha') = 0$ so $\omega\alpha' = 0$ and finally $\omega = 0$. For the surjectivity, it is enough to use Claim 3. In the same way, there is an isomorphism of R' -modules

$$\begin{aligned} e_m \Gamma_\tau e_m &\rightarrow e_l \Gamma_\sigma e_{l+1} \\ \omega &\mapsto b \bar{\xi}(\omega)c. \end{aligned}$$

Thus we get the expected R' -module structure for $e_i \Gamma_\sigma e_j$ except when $i = j \in \{l, l+1\}$ or $i = l+1$ and $j = l$.

Suppose that $i = j = l$. The elements of $e_l \Gamma_\sigma e_l$ are of the form $\lambda e_l + b\omega\alpha$ for $\lambda \in K$ and $\omega \in e_m \Gamma_\sigma e_{l-1}$. We already know that

$$e_m \Gamma_\sigma e_{l-1} \cong u^{\ell_{m,l-1}^\theta} R' + v^{\ell_{m,l-1}^\theta} R'$$

and we get the following isomorphism of R' -modules:

$$\begin{aligned} R' &\cong_K K \oplus u^3(u^{\ell_{m,l-1}^\theta} R' + v^{\ell_{m,l-1}^\theta} R') \rightarrow e_l \Gamma_\sigma e_l \\ (\lambda, u^3 p) &\mapsto \lambda e_l + b\psi_\sigma^{m,l-1}(p)\alpha \end{aligned}$$

(the injectivity comes from $(v)_\tau$ and the injectivity of $\bar{\xi}$).

In the same way, if $i = j = l+1$, there is an isomorphism of R' -modules

$$\begin{aligned} R' &\cong_K K \oplus u^3(u^{\ell_{l+2,m}^\theta} R' + v^{\ell_{l+2,m}^\theta} R') \rightarrow e_{l+1} \Gamma_\sigma e_{l+1} \\ (\lambda, u^3 p) &\mapsto \lambda e_{l+1} + \beta \psi_\sigma^{l+2,m}(p)c. \end{aligned}$$

Finally, suppose that $i = l+1$ and $j = l$. The elements of $e_{l+1} \Gamma_\sigma e_l$ are of the form $\lambda a + \beta\omega\alpha$ and there is an isomorphism of R' -modules

$$\begin{aligned} u^{\ell_{l+1,l}^\theta} R' + v^{\ell_{l+1,l}^\theta} R' &\cong_K K v^{\ell_{l+1,l}^\theta} \oplus u^4 u^{\ell_{l+2,l-1}^\theta} R' \rightarrow e_{l+1} \Gamma_\sigma e_l \\ (\lambda v^{\ell_{l+1,l}^\theta}, u^4 p) &\mapsto \lambda a + \beta \psi_\sigma^{l+2,l-1}(p)\alpha. \end{aligned}$$

Indeed, the only non-immediate thing to check is $aE_l = aC_l$. By induction hypothesis (in particular $(v)_\tau$),

$$\begin{aligned}
aE_l &= abc\beta\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(u_\tau^{\ell_{l+2,l-1}^{\theta,\tau}}\right)\right)\alpha \\
&= \beta\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(Yu_\tau^{\ell_{l+2,l-1}^{\theta,\tau}}\right)\right)\alpha \\
&= \beta\bar{\xi}\left(\psi_\tau^{l+2,l-1}\left(Yv_\tau^{\ell_{l+2,l-1}^{\theta,\tau}}\right)\right)\alpha \\
&= \beta\bar{\xi}\left(\psi_\tau^{l+2,m}\left(v_\tau^{\ell_{l+2,m}^{\theta,\tau}}\right)\right)\bar{\xi}\left(\psi_\tau^{m,l-1}\left(v_\tau^{\ell_{m,l-1}^{\theta,\tau}}\right)\right)\alpha \\
&= ab\bar{\xi}\left(\psi_\tau^{m,l-1}\left(v_\tau^{\ell_{m,l-1}^{\theta,\tau}}\right)\right)\alpha = aC_l.
\end{aligned}$$

For the multiplicativity of the ψ_σ , let us start by noticing, thanks to the beginning of the proof of Lemma 2.15 that for any vertices i, j and k of Q'_τ , we have the following identity:

$$\frac{\ell_{i,j}^\theta + \ell_{j,k}^\theta - \ell_{i,k}^\theta}{n} = \frac{\ell_{i,j}^{\theta,\tau} + \ell_{j,k}^{\theta,\tau} - \ell_{i,k}^{\theta,\tau}}{n-1}.$$

It permits to prove that $\psi_\sigma^{i,j}(w)\psi_\sigma^{j,k}(w') = \psi_\sigma^{i,k}(ww')$ for any $(w, w') \in A_{i,j} \times A_{j,k}$. Indeed this is enough to prove that if w and w' are generators as R' -modules. Suppose for example that $w = u^{\ell_{i,j}^\theta}$ and $w' = u^{\ell_{j,k}^\theta}$. Then we have, using the induction hypothesis,

$$\begin{aligned}
\psi_\sigma^{i,j}\left(u^{\ell_{i,j}^\theta}\right)\psi_\sigma^{j,k}\left(u^{\ell_{j,k}^\theta}\right) &= \bar{\xi}\left(\psi_\tau^{i,j}\left(u_\tau^{\ell_{i,j}^{\theta,\tau}}\right)\psi_\tau^{j,k}\left(u_\tau^{\ell_{j,k}^{\theta,\tau}}\right)\right) \\
&= \bar{\xi}\left(\psi_\tau^{i,k}\left(u_\tau^{\ell_{i,j}^{\theta,\tau} + \ell_{j,k}^{\theta,\tau}}\right)\right) \\
&= \bar{\xi}\left(\phi_\tau\left(X^{\left(\ell_{i,j}^{\theta,\tau} + \ell_{j,k}^{\theta,\tau} - \ell_{i,k}^{\theta,\tau}\right)/2(n-1)}\right)\psi_\tau^{i,k}\left(u_\tau^{\ell_{i,k}^{\theta,\tau}}\right)\right) \\
&= \phi_\sigma\left(X^{\left(\ell_{i,j}^{\theta,\tau} + \ell_{j,k}^{\theta,\tau} - \ell_{i,k}^{\theta,\tau}\right)/2(n-1)}\right)\psi_\sigma^{i,k}\left(u_\tau^{\ell_{i,k}^{\theta,\tau}}\right) \\
&= \phi_\sigma\left(X^{\left(\ell_{i,j}^\theta + \ell_{j,k}^\theta - \ell_{i,k}^\theta\right)/2n}\right)\psi_\sigma^{i,k}\left(u^{\ell_{i,k}^\theta}\right) \\
&= \psi_\sigma^{i,k}\left(u^{\ell_{i,j}^\theta + \ell_{j,k}^\theta}\right).
\end{aligned}$$

The multiplicativity for paths starting or ending at l or $l+1$ can be deduced easily from that. For example, if i and k are vertices of τ .

$$\begin{aligned}
\psi_\sigma^{i,l}\left(u^{\ell_{i,l}^\theta}\right)\psi_\sigma^{l,k}\left(u^{\ell_{l,k}^\theta}\right) &= \psi_\sigma^{i,l-1}\left(u^{\ell_{i,l-1}^\theta}\right)\alpha b\psi_\sigma^{m,k}\left(u^{\ell_{m,k}^\theta}\right) \\
&= \psi_\sigma^{i,l-1}\left(u^{\ell_{i,l-1}^\theta}\right)\psi_\sigma^{l-1,m}(u^3)\psi_\sigma^{m,k}\left(u^{\ell_{m,k}^\theta}\right) \\
&= \psi_\sigma^{i,k}\left(u^{\ell_{i,l-1}^\theta + 3 + \ell_{m,k}^\theta}\right) = \psi_\sigma^{i,k}\left(u^{\ell_{i,l}^\theta + \ell_{l,k}^\theta}\right).
\end{aligned}$$

The last thing to check are the five additional conditions. Points $(i)_\sigma$ to $(iv)_\sigma$ are easy to check and $(v)_\sigma$ is a consequence of them. \square

Theorem 2.30. *There is an isomorphism of R -orders (and R' -algebras):*

$$e_F\Gamma_\sigma e_F = \left[u^{2d(i,j)}R' + v^{2d(i,j)}R'^{\delta_j=i-1}\right]_{i,j \in \llbracket 1,n \rrbracket} \cong \Lambda$$

where the entries of $e_F\Gamma_\sigma e_F$ are R' -submodules of \mathcal{R}' and Λ is defined at (1.2).

For each edge a of σ , the $e_F \Gamma_\sigma e_F$ module

$$M_a := e_F \Gamma_\sigma e_a = {}^t[A_{1,a}, A_{2,a}, \dots, A_{n,a}]$$

is, as a Λ -module, isomorphic to

$${}^t[\overbrace{(Y) \cdots (Y)}^{a_1} \overbrace{(Y^2) \cdots (Y^2)}^{n-a_1}] \quad \text{if } a = (P_{a_1}, *);$$

$${}^t[\overbrace{(X-Y) \cdots (X-Y)}^{a_1} \overbrace{(X^2-Y^2) \cdots (X^2-Y^2)}^{n-a_1}] \quad \text{if } a = (P_{a_1}, \bowtie);$$

$${}^t[\overbrace{R' \cdots R'}^{a_1} \overbrace{(X, Y) \cdots (X, Y)}^{a_2-a_1} \overbrace{(X) \cdots (X)}^{n-a_2}] \quad \text{if } a_1 < a_2;$$

$${}^t[\overbrace{(X, Y) \cdots (X, Y)}^{a_2} \overbrace{(X) \cdots (X)}^{a_1-a_2} \overbrace{(X^2, Y^2) \cdots (X^2, Y^2)}^{n-a_1}] \quad \text{if } a_1 > a_2.$$

Proof. If we order the sides of the polygon in the order $(P_1, P_2), (P_2, P_3), \dots, (P_{n-1}, P_n), (P_n, P_1)$, the first equality is a direct application of Proposition 2.26. Notice that for sides i and j of the polygon, we can rewrite $A_{i,j}$ in the following way:

$$A_{i,j} = u^{2d(i,j)} R' + v^{2d(i,j)} R'^{(\delta_{j=i-1})}.$$

We conjugate by the diagonal matrix with diagonal entries $u^{2d(1,i)}$ for $i \in \llbracket 1, n \rrbracket$ and the matrix we obtain has entries

$$\begin{aligned} & u^{2d(1,i)} \left(u^{2d(i,j)} R' + v^{2d(i,j)} R'^{(\delta_{j=i-1})} \right) u^{-2d(1,j)} \\ &= u^{2d(i,j)-1+2d(1,i)-2d(1,j)} \left(u R' + v R'^{(\delta_{j=i-1})} \right) \\ &= u^{2n\delta_{i \in \llbracket j, 1 \rrbracket} - 1} \left(u R' + v R'^{(\delta_{j=i-1})} \right) = X^{(\delta_{i>j})} R' + X^{-(\delta_{i \leq j})} Y R'^{(\delta_{j=i-1})} \end{aligned}$$

for $i, j \in \llbracket 1, n \rrbracket$. It is Λ .

We obtain the structure of M_a , up to some degree shift, by multiplying on the left by the same diagonal matrix. \square

Remark 2.31. Notice that in Theorem 2.30, the module M_a depends only on the edge a and not on the triangulation σ .

2.5. Counterexample with more than one puncture. In this subsection, we show that we can not expect to generalize these results for polygons with more than one puncture.

We take the triangulation σ of a twice-punctured digon of Figure 2.32. It induces the quiver Q_σ on the right and using the same definition than in Section 2, we get that the natural analogue of Γ_σ is the path algebra of the quiver modulo all obvious commutativity relations. We still call it Γ_σ . Suppose that Γ_σ is a $K[U]$ -order, for U in the center of Γ_σ . We can write $e_1 U = P(\alpha\beta) + a\omega$ where P is a polynomial and $\omega \in e_6 \Gamma_\sigma e_1$. As, for $\ell > 0$, $c(\alpha\beta)^\ell$ is clearly not divisible by c on the right, and as $cU = Uc$, we get that P is a constant polynomial. So, if we denote $\pi : e_1 \Gamma_\sigma e_1 \rightarrow K[\alpha\beta]$ the canonical projection, we get $\pi(U) = P(0) \in K$. So $K[\alpha\beta]$ is not a finitely generated module over $K[\pi(U)] = K$ and therefore $e_1 \Gamma_\sigma e_1$ is not a finitely generated module over $K[U]$. It is a contradiction.

This counterexample is easy to generalize to any polygon with at least two punctures.

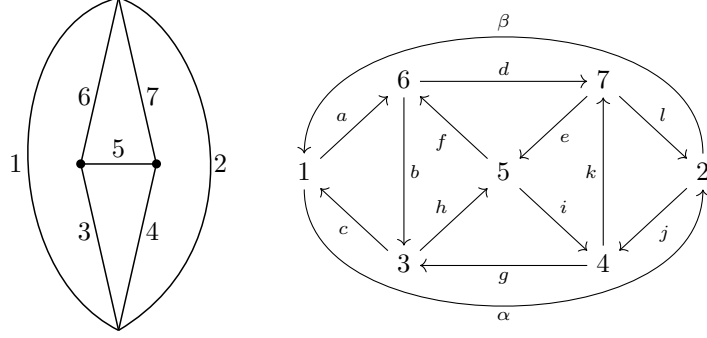
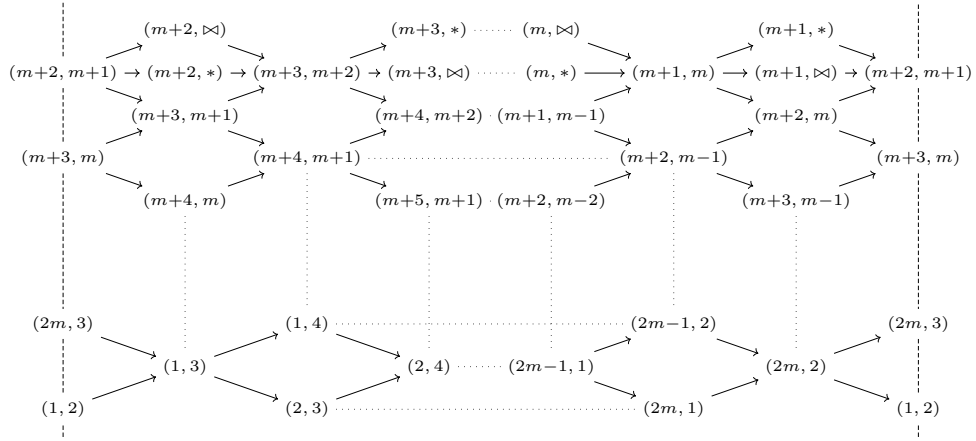


FIGURE 2.32. twice-punctured digon

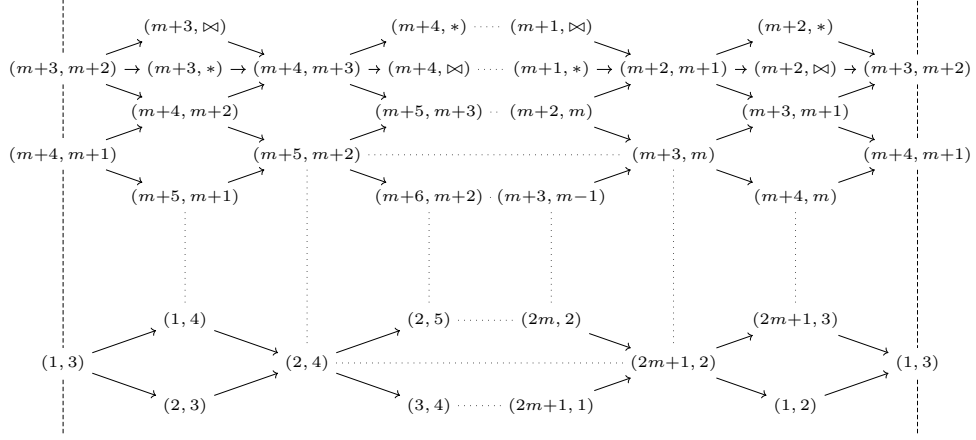
FIGURE 3.1. CMA for $n = 2m$

3. COHEN-MACAULAY MODULES OVER Λ

The aim of this section is to study the representation theory of Λ and its connection to tagged triangulations of the punctured polygon P^* and the cluster category of type D_n . In particular, we classify all Cohen-Macaulay Λ -modules and construct a bijection between the set of the isomorphism classes of all indecomposable Cohen-Macaulay Λ -modules and the set of all sides and tagged arcs of P^* . We then show that the stable category $\underline{\text{CM}} \Lambda$ of Cohen-Macaulay Λ -modules is 2-Calabi-Yau and $\underline{\text{CM}} \Lambda$ is triangle-equivalent with the cluster category of type D_n . To summarize, we will prove that CMA admits the Auslander-Reiten quiver of Figures 3.1, 3.2.

3.1. Classification of Cohen-Macaulay Λ -modules. Let \mathcal{S} be the set of tagged arcs and sides of the once-punctured polygon P^* . In this subsection, we prove the following theorem:

Theorem 3.3. (1) *There is a bijection between \mathcal{S} and the set of isomorphism classes of indecomposable Cohen-Macaulay Λ -modules given by $a \mapsto M_a$ (M_a is defined in Theorem 2.30).*
 (2) *Any Cohen-Macaulay Λ -module is isomorphic to $\bigoplus_{a \in \mathcal{S}} M_a^{l_a}$ for some non-negative integers l_a . Moreover, the l_a are uniquely determined.*

FIGURE 3.2. $\text{CM } \Lambda$ for $n = 2m + 1$

Remark 3.4. Theorem 3.3 shows that Krull-Schmidt-Azumaya property is valid in this case. This is interesting by itself since our base ring $R = K[X]$ is not even local, and in such a case, Krull-Schmidt-Azumaya property usually fails.

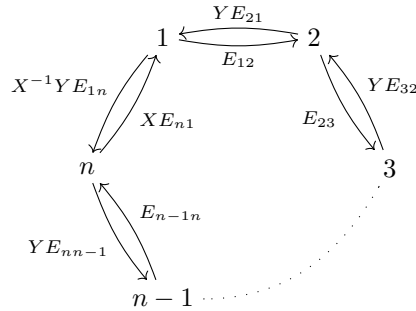
First of all, it is immediate that the M_a are non-isomorphic indecomposable Cohen-Macaulay Λ -modules. To prove this theorem, let us define the following elements of Λ :

$$\alpha_i = E_{i,i+1} \quad \alpha_n = XE_{n,1} \quad \beta_i = YE_{i+1,i} \quad \beta_n = X^{-1}YE_{1,n}.$$

Together with the idempotents E_{ii} , they generate Λ as an R' -algebra and satisfy the relations

$$\begin{cases} \alpha_i \alpha_{i+1} \cdots \alpha_{i-1} = XE_{ii} \\ \beta_{i-1} \beta_{i-2} \cdots \beta_i = Y^{n-1}E_{ii} \\ \alpha_i \beta_i = \beta_{i-1} \alpha_{i-1} = YE_{ii} \end{cases}$$

for $i \in \llbracket 1, n \rrbracket$. In fact, the quiver of Λ is given as follows:



Lemma 3.5. Let $r \in (Y)^{\oplus m}$, $s \in R'^{\oplus p}$ and $t \in (X - Y)^{\oplus q}$ be three vectors such that the ideal I generated by their entries includes the ideal (X, Y) . Then there exists an invertible $(m + p + q) \times (m + p + q)$ matrix

$$G = \begin{pmatrix} A & B & 0 \\ C & D & E \\ 0 & F & G \end{pmatrix}$$

with coefficients in R' where B has coefficients in (Y) and F has coefficients in $(X - Y)$ such that

- either $G \begin{bmatrix} r & s & t \end{bmatrix}$ contains one 1 in its second block and 0 everywhere else;
- either $G \begin{bmatrix} r & s & t \end{bmatrix}$ contains one Y in the first or second block, one $X - Y$ in the second or third block and 0 everywhere else.

Proof. The proof mainly relies on Euclidean algorithm. We can write $r = r'Y$, $s = s' + s''Y$ and $t = t'(X - Y)$ where $r' \in R^{\oplus m}$, $s', s'' \in R^{\oplus p}$ and $t' \in R^{\oplus q}$. Up to applying Euclidean algorithm on the entries of s' and then on the entries of s'' (which is multiplying by an invertible matrix on the left), we can suppose that

$$s = {}^t[Q_1 + Q_2Y \quad Q_3Y]$$

for some $Q_1, Q_2, Q_3 \in R$ (we can ignore 0 entries). With the same method, we can suppose that r has only one (nonzero) entry PY and t has only one (nonzero) entry $S(X - Y)$. Using a sequence of (authorized) matrix multiplications

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ T(X - Y) & 1 \end{bmatrix} \begin{bmatrix} Q_1 + Q_2Y \\ S(X - Y) \end{bmatrix} &= \begin{bmatrix} Q_1 + Q_2Y \\ (S + TQ_1)(X - Y) \end{bmatrix} \\ \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_1 + Q_2Y \\ S(X - Y) \end{bmatrix} &= \begin{bmatrix} (Q_1 + TSX) + (Q_2 - TS)Y \\ S(X - Y) \end{bmatrix}, \end{aligned}$$

thanks to the Euclidean algorithm, we can ensure that $Q_1 = 0$ or $S = 0$. In the same way, we can ensure that $Q_3 = 0$ or $P = 0$. If $Q_1 = 0$, we can ensure that at most one of Q_2, Q_3 and P is nonzero. To summarize, and forgetting about 0 entries, we can ensure to be in one of the following four cases:

- (1) $r = PY, s = 0, t = S(X - Y)$: In this case, by our assumption, $(Y) \oplus (X - Y) = (X, Y) \subset I = (PY, S(X - Y)) = (PY) \oplus (S(X - Y))$, so we deduce that $(Y) = (PY)$ and $(X - Y) = (S(X - Y))$ so up to scalar multiplication $P = S = 1$. This is one of the expected cases.
- (2) $r = 0, s = QY, t = S(X - Y)$: In this case, by the same reasoning than before, up to scalar multiplication $Q = S = 1$. This is one of the expected cases.
- (3) $r = PY, s = Q_1 + Q_2Y, t = 0$: In this case, let us rewrite $s = Q'_1 + Q'_2(X - Y)$ where $Q'_2 = -Q_2$ and $Q'_1 = Q_1 + Q_2X$. Up to a sequence of authorized matrix multiplications

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix} \begin{bmatrix} PY \\ Q'_1 + Q'_2(X - Y) \end{bmatrix} &= \begin{bmatrix} PY \\ (Q'_1 + TPX) + (Q'_2 - TP)(X - Y) \end{bmatrix} \\ \begin{bmatrix} 1 & TY \\ 0 & 1 \end{bmatrix} \begin{bmatrix} PY \\ Q'_1 + Q'_2(X - Y) \end{bmatrix} &= \begin{bmatrix} (P + TQ'_1)Y \\ Q'_1 + Q'_2(X - Y) \end{bmatrix} \end{aligned}$$

we can suppose that $P = 0$ or $Q'_1 = 0$. If $Q'_1 = 0$, we are in the previous case and we can conclude. If $P = 0$, I is a principal ideal containing (X, Y) so $I = R'$ and, up to scalar multiplication, $Q'_1 + Q'_2(X - Y) = 1$. We are again in an expected case.

- (4) $r = 0, s = [Q_1 + Q_2Y \quad Q_3Y], t = 0$: this case is similar to the previous one. \square

Lemma 3.6. *Let $M = R' \oplus M_2$ and N be Cohen-Macaulay R' -modules and $f : M \rightarrow N$, $g, g' : N \rightarrow M$ be morphisms satisfying $gf = X \text{Id}_M$, $fg = X \text{Id}_N$, $g'f = Y \text{Id}_M$ and $fg' = Y \text{Id}_N$. There exists an isomorphism $\phi : N \rightarrow N_1 \oplus N_2$ such that*

$$\phi f = \begin{bmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{bmatrix} \quad ; \quad g\phi^{-1} = \begin{bmatrix} \chi_{11} & \chi_{12} \\ 0 & \chi_{22} \end{bmatrix} \quad ; \quad g'\phi^{-1} = \begin{bmatrix} \chi'_{11} & \chi'_{12} \\ 0 & \chi'_{22} \end{bmatrix}$$

where

- either $N_1 = R'$, $\psi_{11} = 1$, $\chi_{11} = X$ and $\chi'_{11} = Y$;
- or $N_1 = (X, Y)$, $\psi_{11} = X$, χ_{11} is the inclusion and χ'_{11} maps both X and Y to Y .

Proof. Let $f_1 : R' \rightarrow N$ be

$$f_1 = f \circ \begin{bmatrix} \text{Id}_{R'} \\ 0 \end{bmatrix}.$$

As (Y) , R' and $(X - Y)$ are the only isomorphism classes of indecomposable Cohen-Macaulay R' -modules, we can decompose, up to isomorphism of N :

$$N = (Y)^{\oplus m} \oplus R'^{\oplus p} \oplus (X - Y)^{\oplus q} \quad ; \quad f_1 = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

where r is a vector with entries in (Y) , s is a vector with entries in R' and t is a vector with entries in $(X - Y)$. Using $gf = X \text{Id}_N$ and $g'f = Y \text{Id}_N$, we obtain that the ideal generated by the entries of r , s and t contains (X, Y) , so, thanks to Lemma 3.5, up to multiplying f on the left by an invertible matrix and reordering the rows, we can suppose that we are in one of the following cases:

- (1) $N = R' \oplus N_2$ and

$$f = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}.$$

In this case, we can write

$$g = \begin{bmatrix} X & * \\ 0 & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} Y & * \\ 0 & * \end{bmatrix}$$

using the identities $gf = X \text{Id}_N$ and $g'f = Y \text{Id}_N$. We are in the first expected case.

- (2) $N = (Y) \oplus R' \oplus N_2$ and

$$f = \begin{bmatrix} Y & * \\ X - Y & * \\ 0 & * \end{bmatrix}.$$

In this case, we can write

$$g = \begin{bmatrix} \iota_Y & 1 & * \\ * & * & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} \iota_Y & 0 & * \\ * & * & * \end{bmatrix}$$

Up to column operations on g and corresponding row operations on f , we can write

$$f = \begin{bmatrix} Y & * \\ X & 0 \\ 0 & * \end{bmatrix} \quad g = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix} \quad g' = \begin{bmatrix} \iota_Y & 0 & * \\ * & * & * \end{bmatrix}$$

(the 0 on the second column of f comes from $gf = X \text{Id}_N$). It is now easy to see that we cannot get $fg' = Y \text{Id}_M$. So this case is excluded.

- (3) $N = (Y) \oplus (X - Y) \oplus N_2$ and

$$f = \begin{bmatrix} Y & * \\ X - Y & * \\ 0 & * \end{bmatrix}.$$

In this case, we can write

$$g = \begin{bmatrix} \iota_Y & \iota_{X-Y} & * \\ 0 & 0 & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} \iota_Y & 0 & * \\ 0 & 0 & * \end{bmatrix}$$

(once again, we use that (Y) and $(X - Y)$ are in direct sum). Using the equality $(Y) \oplus (X - Y) = (X, Y)$, we are in the second expected case.

(4) $N = R' \oplus R' \oplus N_2$ and

$$f = \begin{bmatrix} Y & * \\ X - Y & * \\ 0 & * \end{bmatrix}.$$

We can write

$$g = \begin{bmatrix} 1 & 1 & * \\ * & * & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} 1 & 0 & * \\ * & * & * \end{bmatrix}$$

As in (ii), using column operations on g , we can rewrite:

$$f = \begin{bmatrix} Y & * \\ X & 0 \\ 0 & * \end{bmatrix} \quad g = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix} \quad g' = \begin{bmatrix} 1 & 0 & * \\ * & * & * \end{bmatrix}$$

and it contradicts $fg' = Y \text{Id}_M$.

(5) $N = R' \oplus (X - Y) \oplus N_2$ and

$$f = \begin{bmatrix} Y & * \\ X - Y & * \\ 0 & * \end{bmatrix}.$$

We can write

$$g = \begin{bmatrix} 1 & \iota_{X-Y} & * \\ * & * & * \end{bmatrix} \quad \text{and} \quad g' = \begin{bmatrix} 1 & 0 & * \\ * & * & * \end{bmatrix}$$

Using column operations on g and g' , we can rewrite:

$$f = \begin{bmatrix} Y & 0 \\ X - Y & * \\ 0 & * \end{bmatrix} \quad g = \begin{bmatrix} 1 & \iota_{X-Y} & * \\ * & * & * \end{bmatrix} \quad g' = \begin{bmatrix} 1 & 0 & 0 \\ * & * & * \end{bmatrix}$$

(the 0 in the second column of f comes from $g'f = Y \text{Id}_N$). We cannot have $fg = X \text{Id}_M$ so this case is excluded. \square

We can easily dualize (over R') the previous lemma:

Lemma 3.7. *Let $M = R' \oplus M_2$ and N be Cohen-Macaulay R' -modules and $f : N \rightarrow M$, $g, g' : M \rightarrow N$ be morphisms satisfying $gf = X \text{Id}_N$, $fg = X \text{Id}_M$, $g'f = Y \text{Id}_N$ and $fg' = Y \text{Id}_M$. There exists an isomorphism $\phi : N \rightarrow N_1 \oplus N_2$ such that*

$$f\phi^{-1} = \begin{bmatrix} \psi_{11} & 0 \\ \psi_{21} & \psi_{22} \end{bmatrix} \quad ; \quad \phi g = \begin{bmatrix} \chi_{11} & 0 \\ \chi_{21} & \chi_{22} \end{bmatrix} \quad ; \quad \phi g' = \begin{bmatrix} \chi'_{11} & 0 \\ \chi'_{21} & \chi'_{22} \end{bmatrix}$$

where

- either $N_1 = R'$, $\psi_{11} = 1$, $\chi_{11} = X$ and $\chi'_{11} = Y$;
- or $N_1 = (X, Y)$, ψ_{11} is the inclusion, $\chi_{11} = X$ and $\chi'_{11} = Y$.

Lemma 3.8. *Let M be a Cohen-Macaulay Λ -module. If M , as an R' -module, has a direct summand isomorphic to R' , then M has a direct summand isomorphic to M_a for some tagged arc or side a of P^* which is not incident to the puncture.*

Proof. For $i \in \llbracket 1, n \rrbracket$, let $M_i = E_{ii}M$. By abuse of notation, we call $\alpha_i : M_{i+1} \rightarrow M_i$ and $\beta_i : M_i \rightarrow M_{i+1}$ the morphisms of R' -modules corresponding to the elements with same names in Λ .

Let $i, j \in \llbracket 1, n \rrbracket$ such that $\alpha_i \alpha_{i+1} \dots \alpha_{j-1} \alpha_j$ has a direct summand isomorphic to

$$R' \xrightarrow{X} R'$$

(such a pair exists as M contains R' as a direct summand, and $\alpha_i \alpha_{i+1} \dots \alpha_{i-1} = X \text{Id}_{M_i}$ for any $i \in \llbracket 1, n \rrbracket$). If $j < i$, note that $\alpha_n \alpha_1$ appears in the previous composition. The number k of factors of this composition is $d(i, j) + 1$. We make the additional assumption that k is as small as possible. Without loss of generality, we can suppose that $i = 1 \leq j$ (the problem is invariant by cyclic permutation). Using Lemma 3.6 for $f = \alpha_j$, $g = \alpha_{j+1} \alpha_{j+2} \dots \alpha_{j-1}$ and $g' = \beta_j$, we get actually that $j > 1$ and we can suppose that $M_{j+1} = R' \oplus M'_{j+1}$, $M_j = (X, Y) \oplus M'_j$ and

$$\alpha_j = \begin{bmatrix} X & \alpha_{j,12} \\ 0 & \alpha_{j,22} \end{bmatrix}$$

(the other possibility of Lemma 3.6 would contradict the minimality of k). Then, we get easily that we can write $M_1 = R' \oplus M'_1$ and

$$\gamma = \alpha_1 \dots \alpha_{j-1} = \begin{bmatrix} \iota_{(X,Y)} & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix}$$

where $\iota_{(X,Y)} : (X, Y) \rightarrow R'$ is the inclusion. Note that by hypothesis

$$\gamma \alpha_j = \begin{bmatrix} X & 0 \\ 0 & * \end{bmatrix}.$$

As all morphisms to R' which are in the radical of CM R' factor through $\iota_{(X,Y)} : (X, Y) \rightarrow R'$, by column operations on γ which do not affect the previous shapes, we can then suppose that one of the following holds:

- $\gamma_{12} = 0$;
- $M'_j = R' \oplus M''_j$ and

$$\gamma = \begin{bmatrix} \iota_{(X,Y)} & 1 & 0 \\ 0 & * & * \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} X & \alpha_{j,12} \\ 0 & * \\ 0 & * \end{bmatrix}$$

then by a column operation on γ , we get

$$\gamma = \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} X & \alpha_{j,12} \\ X & 0 \\ 0 & * \end{bmatrix}$$

(the 0 in the second column of α_j comes from the shape of $\gamma \alpha_j$). But it contradicts the existence of β_j such that $\alpha_j \beta_j = Y \text{Id}_{M_j}$.

Finally, we get the situation

$$\gamma = \begin{bmatrix} \iota_{(X,Y)} & 0 \\ 0 & \gamma_{22} \end{bmatrix} \quad \text{and} \quad \alpha_j = \begin{bmatrix} X & 0 \\ 0 & \alpha_{j,22} \end{bmatrix}$$

Now, using Lemma 3.7 if $j > 2$ permits to suppose that

$$\alpha_1 = \begin{bmatrix} \iota_{(X,Y)} & 0 \\ \alpha_{1,21} & \alpha_{1,22} \end{bmatrix}$$

Then we get easily that

$$\gamma' = \alpha_2 \dots \alpha_{j-1} = \begin{bmatrix} 1 & 0 \\ \gamma'_{21} & \gamma'_{22} \end{bmatrix}.$$

By row operations on γ' (and the corresponding ones on α_1 , we can suppose that

$$\alpha_1 = \begin{bmatrix} \iota_{(X,Y)} & 0 \\ 0 & \alpha_{1,22} \end{bmatrix} \quad \text{and} \quad \gamma' = \begin{bmatrix} 1 & 0 \\ 0 & \gamma'_{22} \end{bmatrix}.$$

We also get easily that

$$\gamma'' = \alpha_{j+1} \dots \alpha_n = \begin{bmatrix} 1 & 0 \\ 0 & \gamma''_{22} \end{bmatrix}.$$

Acting by automorphism on M_3, \dots, M_{j-1} if $j > 3$ and on M_{j+2}, \dots, M_n if $j < n-1$ permits easily to suppose that

$$\alpha_\ell = \begin{bmatrix} 1 & 0 \\ 0 & \alpha_{\ell,22} \end{bmatrix}$$

for any $\ell \in \llbracket 2, j-1 \rrbracket \cup \llbracket j+1, n \rrbracket$. Then, we conclude that M has a direct summand isomorphic to

$${}^t[R' \quad \overbrace{(X, Y) \cdots (X, Y)}^{j-1} \quad \overbrace{(X) \cdots (X)}^{n-j}] \cong M_{(P_1, P_j)}. \quad \square$$

Lemma 3.9. *Let M be a Cohen-Macaulay Λ -module. If M , as an R' -module, has no direct summand isomorphic to R' then M has a direct summand isomorphic to some M_a where a is a tagged arc of P^* incident to the puncture.*

Proof. Denote as before $M_i = E_{ii}M$. As an R' -module, M is a direct sum of copies of (Y) and $(X - Y)$. As there are no morphisms between (Y) and $(X - Y)$, we can suppose that only one of them appear as a summand of M and therefore the matrix coefficients of the α_i are just elements of R . Up to circular permutation, we can suppose that α_n is not invertible. Choose an R -basis $\{e_1, \dots, e_\ell\}$ of M_n such that e_1 is not in the image of α_n . By usual Euclidean algorithm applied on the right of α_n , we can suppose that

$$\alpha_n = \begin{bmatrix} \lambda & 0 \\ * & * \end{bmatrix} \quad \text{and} \quad \alpha_1 \dots \alpha_{n-1} = \begin{bmatrix} \lambda' & 0 \\ * & * \end{bmatrix}$$

As $\lambda'\lambda = X$ and e_1 is not reached by α_n , we can suppose up to scalar change of basis that $\lambda = X$ and $\lambda' = 1$. Hence, by row operations on $\alpha_1 \dots \alpha_{n-1}$, we can suppose that

$$\alpha_n = \begin{bmatrix} X & 0 \\ 0 & * \end{bmatrix} \quad \text{and} \quad \alpha_1 \dots \alpha_{n-1} = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$$

(the lower left 0 of α_n comes from $\alpha_n \alpha_1 \alpha_2 \dots \alpha_{n-1} = X \text{Id}_{M_n}$). Therefore, by changes of basis of M_2, \dots, M_{n-1} , we can suppose that

$$\alpha_\ell = \begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix}$$

for $\ell \in \llbracket 1, n-1 \rrbracket$. Finally, M as a direct summand isomorphic to

$${}^t[\overbrace{(Y) \cdots (Y)}^n] \cong M_{(P_n, *)}$$

$$\text{or } {}^t[\overbrace{(X - Y) \cdots (X - Y)}^n] \cong M_{(P_n, \bowtie)}. \quad \square$$

Proof of Theorem 3.3. First of all, thanks to Lemmas 3.8 and 3.9, any Cohen-Macaulay Λ -module can be decomposed as expected.

For the uniqueness of the decomposition, we need to use Proposition 3.10 (notice that we do not use Theorem 3.3 in its proof). The endomorphism algebra of M_a is isomorphic to R' if a is not incident to the puncture and isomorphic to R if a is incident to the puncture. Moreover, any endomorphism factorizing through another indecomposable is in the ideal (X, Y) in the first case and (X) in the second case. Thus, if we denote $\hat{\Lambda} = K[[X]] \otimes_R \Lambda$, and consider the functor $K[[X]] \otimes_R - : \text{CM } \Lambda \rightarrow \text{CM } \hat{\Lambda}$, non-isomorphic indecomposable objects are mapped to non-isomorphic objects, which are also indecomposable. Moreover, the endomorphism rings of the objects $K[[X]] \otimes_R M_a$ are local so we get the uniqueness of the decomposition of objects in the essential image of the functor $K[[X]] \otimes_R -$. It permits to conclude. \square

3.2. Homological structure of $\text{CM } \Lambda$. The aim of this subsection is to compute spaces of morphisms and extensions in the category $\text{CM } \Lambda \cong \text{CM } \Lambda'$ where $\Lambda' = e_F \Gamma_\sigma e_F$. For convenience of notation, we will work on Λ' . Notice that the definitions of $a \vdash b$ and $A_{a,b}$ given for two tagged arcs a and b before Proposition 2.26 make sense even when a and b are not compatible (there are cases where $a \vdash b$ other than the one depicted there).

Proposition 3.10. *Let a and b be two tagged arcs or sides of P^* . Following the notation of Proposition 2.26, we have the following isomorphism:*

$$\text{Hom}_{\Lambda'}(M_a, M_b) \cong A_{a,b}.$$

Moreover, these morphisms are realized by right multiplication in \mathcal{R}' and therefore, the composition of morphisms corresponds to the multiplication in \mathcal{R}' .

Proof. First of all, for $i \in \llbracket 1, n \rrbracket$, using Theorem 2.30, recall that $E_{ii}M_a \cong A_{i,a}$ and $E_{ii}M_b \cong A_{i,b}$ (in a compatible way with the Λ' -module structure). Thus, we know that for any i , $\text{Hom}_{R'}(A_{i,a}, A_{i,b})$ can be realized as an R' -submodule of \mathcal{R}' through multiplication. Namely,

$A_{i,a} \backslash A_{i,b}$	$u^{j'-1}vR'$	$u^{j'-1}(u-v)R'$	$u^{j'}R'$
$u^{j-1}vR'$	$u^{j'-j-1}vR'$	0	$u^{j'-j+2n-1}vR'$
$u^{j-1}(u-v)R'$	0	$u^{j'-j-1}(u-v)R'$	$u^{j'-j+2n-1}(u-v)R'$
u^jR'	$u^{j'-j-1}vR'$	$u^{j'-j-1}(u-v)R'$	$u^{j'-j}R'$

where $j = \ell_{i,a}^\theta$ and $j' = \ell_{i,b}^\theta$ (the only other kind of $A_{i,a}$ or $A_{i,b}$ which can appear is $u^jR' + u^{j-1}vR' = u^{j-1}((u-v)R' \oplus vR')$ which can be realized as the direct sum of the two first rows, and two first columns. In any of these cases, the sum is direct inside \mathcal{R}').

If $f \in \text{Hom}_{\Lambda'}(M_a, M_b)$, let $f_i \in \text{Hom}_{R'}(A_{i,a}, A_{i,b})$ be its i -th component. As $u^2 E_{i,i+1} \in \Lambda'$, for any $m \in M_a$, $f(u^2 E_{i,i+1} m) = u^2 E_{i,i+1} f(m)$ holds. It can be rewritten as $f_i(u^2 m_{i+1}) = u^2 f_{i+1}(m_{i+1})$ or again $f_i u^2 m_{i+1} = u^2 f_{i+1} m_{i+1}$ if f_i, f_{i+1} are considered as elements of \mathcal{R}' . As u^2 is invertible in \mathcal{R}' , we get $f_i m_{i+1} = f_{i+1} m_{i+1}$. This is true for any $m_{i+1} \in A_{i+1,a}$ so $f_i - f_{i+1}$ is in the annihilator of $A_{i+1,a}$. Looking at Theorem 2.30, the annihilators of $A_{i+1,a}$ and $A_{i,a}$ are the same and included in

- $(u-v)\mathcal{R}'$ if a is incident to the puncture and plain;
- $v\mathcal{R}'$ if a is incident to the puncture and notched;
- 0 if a is not incident to the puncture.

Moreover, looking at the previous table,

- $\text{Hom}_{R'}(A_{i,a}, A_{i,b}) + \text{Hom}_{R'}(A_{i+1,a}, A_{i+1,b}) \subset v\mathcal{R}'$ if a is incident to the puncture and plain;
- $\text{Hom}_{R'}(A_{i,a}, A_{i,b}) + \text{Hom}_{R'}(A_{i+1,a}, A_{i+1,b}) \subset (u-v)\mathcal{R}'$ if a is incident to the puncture and notched.

so $\text{Hom}_{R'}(A_{i,a}, A_{i,b}) + \text{Hom}_{R'}(A_{i+1,a}, A_{i+1,b})$ intersects the annihilator of $A_{i,a}$ at 0 and we obtain that $f_i = f_{i+1}$.

Finally, we get

$$\text{Hom}_{\Lambda'}(M_a, M_b) = \bigcap_{i=1}^n \text{Hom}_{R'}(A_{i,a}, A_{i,b})$$

as R' -submodules of \mathcal{R}' .

(a) Suppose now that none of a and b is incident to the puncture. For $i \in \llbracket 1, n \rrbracket$, we have

$$A_{i,a} = \begin{cases} u^{\ell_{(P_i, P_{i+1}), a}^\theta} R' & \text{if } (P_i, P_{i+1}) \nVdash a \\ u^{\ell_{(P_i, P_{i+1}), a}^\theta} (R' + u^{-1}vR') & \text{if } (P_i, P_{i+1}) \vdash a \end{cases}$$

and

$$A_{i,b} = \begin{cases} u^{\ell_{(P_i, P_{i+1}), b}^\theta} R' & \text{if } (P_i, P_{i+1}) \nVdash b \\ u^{\ell_{(P_i, P_{i+1}), b}^\theta} (R' + u^{-1}vR') & \text{if } (P_i, P_{i+1}) \vdash b. \end{cases}$$

Therefore, we obtain

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{(P_i, P_{i+1}), b}^\theta - \ell_{(P_i, P_{i+1}), a}^\theta} R'$$

if $(P_i, P_{i+1}) \nVdash a$ and $(P_i, P_{i+1}) \nVdash b$,

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{(P_i, P_{i+1}), b}^\theta - \ell_{(P_i, P_{i+1}), a}^\theta + 2n} (R' + u^{-1}vR')$$

if $(P_i, P_{i+1}) \vdash a$ and $(P_i, P_{i+1}) \nVdash b$,

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{(P_i, P_{i+1}), b}^\theta - \ell_{(P_i, P_{i+1}), a}^\theta} (R' + u^{-1}vR')$$

if $(P_i, P_{i+1}) \vdash b$.

Using Lemma 2.15, we obtain

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^\theta - 2n(\delta_{b_2 \in \llbracket b_1, a_2 \rrbracket} \delta_{a_1 \in \llbracket a_2, b_1 \rrbracket} + \delta_{i \in \llbracket a_1, b_1 \rrbracket})} R'$$

if $(P_i, P_{i+1}) \nVdash a$ and $(P_i, P_{i+1}) \nVdash b$,

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^\theta - 2n(\delta_{b_2 \in \llbracket b_1, a_2 \rrbracket} \delta_{a_1 \in \llbracket a_2, b_1 \rrbracket} + \delta_{i \in \llbracket a_1, b_1 \rrbracket} - 1)} (R' + u^{-1}vR')$$

if $(P_i, P_{i+1}) \vdash a$ and $(P_i, P_{i+1}) \nVdash b$,

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^\theta - 2n(\delta_{b_2 \in \llbracket b_1, a_2 \rrbracket} \delta_{a_1 \in \llbracket a_2, b_1 \rrbracket} + \delta_{i \in \llbracket a_1, b_1 \rrbracket})} (R' + u^{-1}vR')$$

if $(P_i, P_{i+1}) \vdash b$.

Notice that $(P_i, P_{i+1}) \vdash a$ if and only if $i \in \llbracket a_1, a_2 \rrbracket$.

(a-1) Suppose that $a \nVdash b$. It means that $a_2 \notin \llbracket b_2, a_1 \rrbracket$ and $b_1 \notin \llbracket b_2, a_1 \rrbracket$. In this case, $\delta_{b_2 \in \llbracket b_1, a_2 \rrbracket} \delta_{a_1 \in \llbracket a_2, b_1 \rrbracket} = 0$. Taking $i = a_1$, we have $(P_i, P_{i+1}) \nVdash a$ and $(P_i, P_{i+1}) \nVdash b$ and an easy computation gives $\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^\theta} R'$. The only way to get a smaller module would be in the case $(P_i, P_{i+1}) \vdash a$ and $(P_i, P_{i+1}) \nVdash b$ that is $i \in \llbracket a_1, a_2 \rrbracket \cap \llbracket b_2, b_1 \rrbracket$. With the current hypotheses, we get that $\llbracket a_1, a_2 \rrbracket \cap \llbracket b_2, b_1 \rrbracket \subset \llbracket a_1, b_1 \rrbracket$, so actually, we cannot get a smaller module.

(a-2) Suppose now that $a \vdash b$. It means that $a_2 \in \llbracket b_2, a_1 \rrbracket$ or $b_1 \in \llbracket b_2, a_1 \rrbracket$. Let us consider two cases:

- When $b_2 \in \llbracket b_1, a_2 \rrbracket$ and $a_1 \in \llbracket a_2, b_1 \rrbracket$. Taking $i = b_2$, we have $(P_i, P_{i+1}) \vdash a$ and $(P_i, P_{i+1}) \nVdash b$ and an easy computation gives

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^\theta} (R' + u^{-1}vR').$$

Thanks to the term $\delta_{b_2 \in \llbracket b_1, a_2 \rrbracket} \delta_{a_1 \in \llbracket a_2, b_1 \rrbracket}$, submodules that appear for any other i are bigger.

- When $b_2 \notin \llbracket b_1, a_2 \rrbracket$ or $a_1 \notin \llbracket a_2, b_1 \rrbracket$. In this case, as $b_1 \neq b_2$, we get that in fact $b_1 \in \llbracket b_2, a_1 \rrbracket$. Taking $i = a_1$, we obtain $(P_i, P_{i+1}) \vdash b$ and

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^\theta} (R' + u^{-1}vR').$$

Other submodules are bigger as, if $(P_i, P_{i+1}) \nVdash b$, that is $i \in \llbracket b_2, b_1 \rrbracket$, we would have $i \in \llbracket a_1, b_1 \rrbracket$.

We finished the case where none of a and b are incident to the puncture.

(b) Suppose now that both a and b are incident to the puncture. For $i \in \llbracket 1, n \rrbracket$, we have

$$A_{i,a} = \begin{cases} u^{\ell_{(P_i, P_{i+1}), a}^{-1}} v R' & \text{if } a \text{ is plain} \\ u^{\ell_{(P_i, P_{i+1}), a}^{-1}} (u - v) R' & \text{if } a \text{ is notched} \end{cases}$$

and

$$A_{i,b} = \begin{cases} u^{\ell_{(P_i, P_{i+1}), b}^{-1}} v R' & \text{if } b \text{ is plain} \\ u^{\ell_{(P_i, P_{i+1}), b}^{-1}} (u - v) R' & \text{if } b \text{ is notched.} \end{cases}$$

As there are no morphisms if the tags are different, we can suppose that both a and b are plain, and we obtain, for any $i \in \llbracket 1, n \rrbracket$,

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{(P_i, P_{i+1}), b} - \ell_{(P_i, P_{i+1}), a}^{-1}} v R'.$$

Using Lemma 2.15 and the fact that $b_1 = b_2$, we deduce that

$$\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b} - 2n\delta_{i \in \llbracket a_1, b_1 \rrbracket} - 1} v R'$$

and for $i = a_1$, $\text{Hom}_{R'}(A_{i,a}, A_{i,b}) = u^{\ell_{a,b}^{-1}} v R'$ which is of course the smallest possible.

(c) Suppose now that a is incident to the puncture and b is not. Without loss of generality, we can suppose a is plain. We have $(X - Y)M_a = 0$. Therefore, for any $f \in \text{Hom}_{\Lambda'}(M_a, M_b)$, $(X - Y)\text{Im } f = 0$. Notice now that

$$M'_b = \{m \in M_b \mid (X - Y)m = 0\}$$

satisfies

$$\begin{aligned} E_{ii}M'_b &= \begin{cases} u^{\ell_{(P_i, P_{i+1}), b} + 2n - 1} v R' & \text{if } (P_i, P_{i+1}) \nmid b \\ u^{\ell_{(P_i, P_{i+1}), b}^{-1}} v R' & \text{if } (P_i, P_{i+1}) \vdash b \end{cases} \\ &= u^{\ell_{(P_i, P_{i+1}), b} + 2n\delta_{i \in \llbracket b_2, b_1 \rrbracket} - 1} v R'. \end{aligned}$$

Let $b' = (P_{b_2}, *)$. Thanks to Lemma 2.15, we can rewrite

$$\begin{aligned} &\ell_{(P_i, P_{i+1}), b} + 2n\delta_{i \in \llbracket b_2, b_1 \rrbracket} \\ &= \ell_{(P_i, P_{i+1}), b'} + \ell_{b', b} - 2n\delta_{i \in \llbracket b_2, b_1 \rrbracket} + 2n\delta_{i \in \llbracket b_2, b_1 \rrbracket} = \ell_{(P_i, P_{i+1}), b'} + \ell_{b', b}. \end{aligned}$$

Thus $M'_b = u^{\ell_{b', b}} M_{b'}$ and

$$\text{Hom}_{\Lambda'}(M_a, M_b) = u^{\ell_{b', b}} \text{Hom}_{\Lambda'}(M_a, M_{b'}) = u^{\ell_{b', b} + \ell_{a, b'}^{-1}} v R'$$

and we have (using $a_1 = a_2$),

$$\begin{aligned} \ell_{b', b} + \ell_{a, b'} &= d(b_2, b_1) + 2d(a_1, b_2) \\ &= d(a_1, b_1) + n\delta_{b_2 \in \llbracket b_1, a_1 \rrbracket} + d(a_1, b_2) = \ell_{a, b}^{\theta} \end{aligned}$$

which concludes this case.

(d) Finally, suppose that b is incident to the puncture and a is not. Without loss of generality, we can suppose b is plain. As $(X - Y)M_b = 0$,

$$\text{Hom}_{\Lambda'}(M_a, M_b) = \text{Hom}_{\Lambda'}(M'_a, M_b)$$

where $M'_a = M_a / (X - Y)M_a$. Using the same idea as before,

$$E_{ii}M'_a = u^{\ell_{(P_i, P_{i+1}), a}^{-1}} v R'$$

and thanks to Lemma 2.15, if $a' = (P_{a_1}, *)$,

$$\begin{aligned} \ell_{(P_i, P_{i+1}), a}^{\theta} &= \ell_{(P_i, P_{i+1}), a'}^{\theta} + \ell_{a', a}^{\theta} - 2n(\delta_{a_2 \in \llbracket a_1, a_1 \rrbracket} \delta_{a_1 \in \llbracket a_1, a_1 \rrbracket} + \delta_{i \in \llbracket a_1, a_1 \rrbracket}) \\ &= \ell_{(P_i, P_{i+1}), a'}^{\theta} + \ell_{a', a}^{\theta} - 2n \end{aligned}$$

and therefore, $M'_a = u^{\ell_{a',a}^\theta - 2n} M_{a'}$. Thus

$$\text{Hom}_{\Lambda'}(M_a, M_b) = u^{2n - \ell_{a',a}^\theta} \text{Hom}_{\Lambda'}(M_{a'}, M_b) = u^{2n - \ell_{a',a}^\theta + \ell_{a',b}^\theta - 1} v R'$$

and, using $b_1 = b_2$,

$$\begin{aligned} 2n - \ell_{a',a}^\theta + \ell_{a',b}^\theta &= 2n - (d(a_1, a_2) + n) + 2d(a_1, b_1) \\ &= 2n - (n - d(a_2, a_1) + n) + 2d(a_1, b_1) \\ &= d(a_1, b_1) + d(a_2, b_1) + n\delta_{a_1 \in \llbracket b_1, a_2 \rrbracket} = \ell_{a,b}^\theta. \end{aligned}$$

It concludes the proof. \square

Proposition 3.11. *Let a and b be two tagged arcs or sides. Let M_a and M_b be the corresponding indecomposable Λ' -modules. We have the following isomorphism of graded R' -modules:*

- $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) = 0$ if a and b are both incident to the puncture with different tags;
- $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) = u^{\ell_{a,b}^\theta} (R'/(X, Y))^{\oplus \varepsilon}$, where

$$\varepsilon = \delta_{a_1 - 1 \in \llbracket b_1, b_2 \rrbracket} \delta_{b_2 + 1 \in \llbracket a_1, a_2 \rrbracket} = \delta_{a_2 - 1 \in \llbracket b_1, b_2 \rrbracket} \delta_{b_1 + 1 \in \llbracket a_1, a_2 \rrbracket}$$

if either a and b are both incident to the puncture with the same tag, or exactly one of them is incident to the puncture.

- $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) = u^{\ell_{a,b}^\theta} (R'/(X, Y))^{\oplus \varepsilon}$, where

$$\varepsilon = \delta_{a_1 - 1 \in \llbracket b_1, b_2 \rrbracket} \delta_{b_2 + 1 \in \llbracket a_1, a_2 \rrbracket} + \delta_{a_2 - 1 \in \llbracket b_1, b_2 \rrbracket} \delta_{b_1 + 1 \in \llbracket a_1, a_2 \rrbracket}$$

if none of a and b is incident to the puncture.

Proof. (a) Suppose first that none of a and b is incident to the puncture. For any $i \in \llbracket 1, n \rrbracket$, let P_i be the projective module corresponding to the arc (P_i, P_{i+1}) of the polygon. Thanks to Proposition 3.10, we have

$$\text{Hom}_{\Lambda'}(M_a, P_i) = \begin{cases} u^{\ell_{a,(P_i, P_{i+1})}^\theta} R', & \text{if } a \nmid (P_i, P_{i+1}); \\ u^{\ell_{a,(P_i, P_{i+1})}^\theta} (R' + u^{-1}vR'), & \text{if } a \vdash (P_i, P_{i+1}); \end{cases}$$

and

$$\text{Hom}_{\Lambda'}(P_i, M_b) = \begin{cases} u^{\ell_{(P_i, P_{i+1}), b}^\theta} R', & \text{if } (P_i, P_{i+1}) \nmid b; \\ u^{\ell_{(P_i, P_{i+1}), b}^\theta} (R' + u^{-1}vR'), & \text{if } (P_i, P_{i+1}) \vdash b. \end{cases}$$

As a consequence, we get

$$\text{Hom}_{\Lambda'}(P_i, M_b) \circ \text{Hom}_{\Lambda'}(M_a, P_i) = u^{\ell_{a,(P_i, P_{i+1})}^\theta + \ell_{(P_i, P_{i+1}), b}^\theta} R'$$

if $a \nmid (P_i, P_{i+1})$ and $(P_i, P_{i+1}) \nmid b$ and

$$\text{Hom}_{\Lambda'}(P_i, M_b) \circ \text{Hom}_{\Lambda'}(M_a, P_i) = u^{\ell_{a,(P_i, P_{i+1})}^\theta + \ell_{(P_i, P_{i+1}), b}^\theta} (R' + u^{-1}vR')$$

if $a \vdash (P_i, P_{i+1})$ or $(P_i, P_{i+1}) \vdash b$.

Using Lemma 2.15,

$$\ell_{a,(P_i, P_{i+1})}^\theta + \ell_{(P_i, P_{i+1}), b}^\theta = \ell_{a,b}^\theta + 2n(\delta_{a_1 \in \llbracket a_2 - 1, b_1 \rrbracket} \delta_{b_2 \in \llbracket a_2 - 1, b_1 \rrbracket} + \delta_{i \in \llbracket b_1, a_2 - 1 \rrbracket})$$

The minimum is reached for $i \in \llbracket a_2 - 1, b_1 \rrbracket$ and is

$$\ell_{a,b}^\theta + 2n\delta_{b_2 \in \llbracket a_2 - 1, b_1 \rrbracket} \delta_{a_1 \in \llbracket a_2 - 1, b_1 \rrbracket} = \ell_{a,b}^\theta + 2n\delta_{a_2 - 1 \in \llbracket b_1, b_2 \rrbracket} \delta_{b_1 + 1 \in \llbracket a_1, a_2 \rrbracket}.$$

Recall now that $a \vdash (P_i, P_{i+1})$ if and only if $a_2 \in \llbracket i + 1, a_1 \rrbracket$ if and only if $i \in \llbracket a_1 - 1, a_2 - 1 \rrbracket$ and $(P_i, P_{i+1}) \vdash b$ if and only if $b_1 \in \llbracket b_2, i \rrbracket$ if and only if $i \in \llbracket b_1, b_2 \rrbracket$.

So $a \vdash (P_i, P_{i+1})$ or $(P_i, P_{i+1}) \vdash b$ if and only if $i \in \llbracket a_1 - 1, a_2 - 1 \rrbracket \cup \llbracket b_1, b_2 \rrbracket$. If $\llbracket a_1 - 1, a_2 - 1 \rrbracket \cup \llbracket b_1, b_2 \rrbracket$ intersects $\llbracket a_2 - 1, b_1 \rrbracket$, we deduce that

$$\begin{aligned} \mathcal{P}(M_a, M_b) &= \sum_{i=1}^n \text{Hom}_{\Lambda'}(P_i, M_b) \circ \text{Hom}_{\Lambda'}(M_a, P_i) \\ &= u^{\ell_{a,b}^\theta + 2n\delta_{a_2-1 \in \llbracket b_1, b_2 \rrbracket} \llbracket \delta_{b_1+1 \in \llbracket a_1, a_2 \rrbracket} \rrbracket} (R' + u^{-1}vR') \end{aligned}$$

and otherwise

$$\begin{aligned} \mathcal{P}(M_a, M_b) &= \sum_{i=1}^n \text{Hom}_{\Lambda'}(P_i, M_b) \circ \text{Hom}_{\Lambda'}(M_a, P_i) \\ &= u^{\ell_{a,b}^\theta + 2n\delta_{a_2-1 \in \llbracket b_1, b_2 \rrbracket} \llbracket \delta_{b_1+1 \in \llbracket a_1, a_2 \rrbracket} \rrbracket} R'. \end{aligned}$$

Notice that $\llbracket a_1 - 1, a_2 - 1 \rrbracket \cup \llbracket b_1, b_2 \rrbracket$ intersects $\llbracket a_2 - 1, b_1 \rrbracket$ if and only if $b_1 + 1 \in \llbracket a_1, a_2 \rrbracket$ or $a_2 - 1 \in \llbracket b_1, b_2 \rrbracket$, if and only if $b_1 + 1 \in \llbracket a_1, a_2 \rrbracket$ or $a_2 - 1 \in \llbracket b_1, b_2 \rrbracket$ or $a_1 = b_1 + 1$ or $a_2 = b_2 + 1$.

Then, we can simplify $\mathcal{P}(M_a, M_b)$ in the following way:

Case 1: $\mathcal{P}(M_a, M_b) = u^{\ell_{a,b}^\theta + 2n} (R' + u^{-1}vR')$, if $b_1 + 1 \in \llbracket a_1, a_2 \rrbracket$ and $a_2 - 1 \in \llbracket b_1, b_2 \rrbracket$;
Case 2: $\mathcal{P}(M_a, M_b) = u^{\ell_{a,b}^\theta} R'$, if $b_1 + 1 \notin \llbracket a_1, a_2 \rrbracket$ and $a_2 - 1 \notin \llbracket b_1, b_2 \rrbracket$;
Case 3: $\mathcal{P}(M_a, M_b) = u^{\ell_{a,b}^\theta} (R' + u^{-1}vR')$, else.

Recall also that

$$\mathcal{P}(M_a, M_b) \subset \text{Hom}_{\Lambda'}(M_a, M_b) = \begin{cases} u^{\ell_{a,b}^\theta} R' & \text{if } a \nvdash b \\ u^{\ell_{a,b}^\theta} (R' + u^{-1}vR') & \text{if } a \vdash b \end{cases}$$

and therefore, in case 3, we will always get $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) = 0$. In case 1, if $a \nvdash b$, we get $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell_{a,b}^\theta} R' / (X, Y)$ (as graded R' -modules); if $a \vdash b$, we get $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell_{a,b}^\theta} (R' / (X, Y) \oplus R' / (X, Y))$. In case 2, if $a \nvdash b$, we get $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) = 0$; if $a \vdash b$, we get $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell_{a,b}^\theta} R' / (X, Y)$.

Notice that $a \vdash b$ if and only if $a_1 - 1 \in \llbracket b_1, b_2 \rrbracket$ or $b_2 + 1 \in \llbracket a_1, a_2 \rrbracket$. Then, by an easy case by case analysis, it concludes the case where none of a and b are incident to the puncture.

(b) Suppose now that at least one of a and b is incident to the puncture. Without loss of generality, we can suppose that no notched tag appears. An easy computation shows us that, in any case,

$$\text{Hom}_{\Lambda'}(P_i, M_b) \circ \text{Hom}_{\Lambda'}(M_a, P_i) = u^{\ell_{a,(P_i, P_{i+1})}^\theta + \ell_{(P_i, P_{i+1}), b}^\theta} vR'.$$

Using Lemma 2.15,

$$\ell_{a,(P_i, P_{i+1})}^\theta + \ell_{(P_i, P_{i+1}), b}^\theta = \ell_{a,b}^\theta + 2n (\delta_{a_1 \in \llbracket a_2-1, b_1 \rrbracket} \delta_{b_2 \in \llbracket a_2-1, b_1 \rrbracket} + \delta_{i \in \llbracket b_1, a_2-1 \rrbracket})$$

which gives, as before, that the minimum is reached for $i \in \llbracket a_2 - 1, b_1 \rrbracket$ and is

$$\ell_{a,b}^\theta + 2n\delta_{a_2-1 \in \llbracket b_1, b_2 \rrbracket} \llbracket \delta_{b_1+1 \in \llbracket a_1, a_2 \rrbracket} \rrbracket.$$

Therefore, $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) \cong u^{\ell_{a,b}^\theta} R' / (X, Y)$ if $a_2 - 1 \in \llbracket b_1, b_2 \rrbracket$ and $b_1 + 1 \in \llbracket a_1, a_2 \rrbracket$ and $\underline{\text{Hom}}_{\Lambda'}(M_a, M_b) = 0$ else. \square

Proposition 3.12. *The category $\text{CM } \Lambda'$ admits the following Auslander-Reiten sequences:*

$$\begin{aligned} 0 \rightarrow M_{(P_i, P_j)} \xrightarrow{\begin{bmatrix} u \\ u \end{bmatrix}} M_{(P_{i+1}, P_j)} \oplus M_{(P_i, P_{j+1})} \xrightarrow{\begin{bmatrix} -u & u \end{bmatrix}} M_{(P_{i+1}, P_{j+1})} \rightarrow 0 \quad (j \neq i, i+1); \\ 0 \rightarrow M_{(P_i, *)} \xrightarrow{v} M_{(P_{i+1}, P_i)} \xrightarrow{u-v} M_{(P_{i+1}, \bowtie)} \rightarrow 0; \end{aligned}$$

$$0 \rightarrow M_{(P_i, \bowtie)} \xrightarrow{u-v} M_{(P_{i+1}, P_i)} \xrightarrow{v} M_{(P_{i+1}, *)} \rightarrow 0.$$

Notice that $M_{(P_i, P_i)}$, if it appears, has to be interpreted as $M_{(P_i, *)} \oplus M_{(P_i, \bowtie)}$. Thus, $\text{CM } \Lambda'$ admits an Auslander-Reiten translation τ defined by

$$\begin{aligned} \tau(M_{(P_i, P_j)}) &= M_{(P_{i-1}, P_{j-1})} \quad \text{if } j \neq i, i+1; \\ \tau(M_{(P_i, *)}) &= M_{(P_{i-1}, \bowtie)}; \\ \tau(M_{(P_i, \bowtie)}) &= M_{(P_{i-1}, *)}. \end{aligned}$$

Proof. (a) Consider the first case. Let a be a side or an arc of P^* which is not incident to the puncture or a formal sum $(P_{a_1}, *) \oplus (P_{a_1}, \bowtie)$ and $f : M_{(P_i, P_j)} \rightarrow M_a$ be a morphism which is not a split monomorphism. According to Proposition 3.10, the degree $\deg(f)$ of f is at least

$$\ell_{(P_i, P_j), a}^\theta + 2n\delta_{a=(P_i, P_j)}.$$

Moreover, using the beginning of the proof of Lemma 2.15, we get the equalities

$$\begin{aligned} &\ell_{(P_i, P_j), (P_{i+1}, P_j)}^\theta + \ell_{(P_{i+1}, P_j), a}^\theta - \ell_{(P_i, P_j), a}^\theta \\ &= n(\delta_{i=a_1} + 0 + 0 + |\delta_{i+1 \in \llbracket a_1, j \rrbracket} - \delta_{a_2 \in \llbracket a_1, j \rrbracket}| - |\delta_{i \in \llbracket a_1, j \rrbracket} - \delta_{a_2 \in \llbracket a_1, j \rrbracket}|) \\ &= n(\delta_{i=a_1} + (\delta_{a_2 \in \llbracket j, a_1 \rrbracket} - \delta_{a_2 \in \llbracket a_1, j \rrbracket})(\delta_{i+1 \in \llbracket a_1, j \rrbracket} - \delta_{i \in \llbracket a_1, j \rrbracket})) \\ &= n(\delta_{i=a_1} + (2\delta_{a_2 \in \llbracket j, a_1 \rrbracket} - 1)\delta_{i=a_1}) = 2n\delta_{i=a_1}\delta_{a_2 \in \llbracket j, a_1 \rrbracket} \end{aligned}$$

so $\deg(f) \geq \ell_{(P_i, P_j), (P_{i+1}, P_j)}^\theta + \ell_{(P_{i+1}, P_j), a}^\theta + 2n(\delta_{a=(P_i, P_j)} - \delta_{i=a_1}\delta_{a_2 \in \llbracket j, a_1 \rrbracket})$ and

$$\begin{aligned} &\ell_{(P_i, P_j), (P_i, P_{j+1})}^\theta + \ell_{(P_i, P_{j+1}), a}^\theta - \ell_{(P_i, P_j), a}^\theta \\ &= n(0 + \delta_{j=a_2} + 0 + |\delta_{i \in \llbracket a_1, j+1 \rrbracket} - \delta_{a_2 \in \llbracket a_1, j+1 \rrbracket}| - |\delta_{i \in \llbracket a_1, j \rrbracket} - \delta_{a_2 \in \llbracket a_1, j \rrbracket}|) \\ &= n(\delta_{j=a_2} - (\delta_{i \in \llbracket a_1, j \rrbracket} - \delta_{i \in \llbracket j, a_1 \rrbracket})\delta_{j=a_2}) = 2n\delta_{j=a_2}\delta_{i \in \llbracket j, a_1 \rrbracket} \end{aligned}$$

so $\deg(f) \geq \ell_{(P_i, P_j), (P_i, P_{j+1})}^\theta + \ell_{(P_i, P_{j+1}), a}^\theta + 2n(\delta_{a=(P_i, P_j)} - \delta_{j=a_2}\delta_{i \in \llbracket j, a_1 \rrbracket})$. As at least one of $\delta_{a=(P_i, P_j)} - \delta_{i=a_1}\delta_{a_2 \in \llbracket j, a_1 \rrbracket}$ and $\delta_{a=(P_i, P_j)} - \delta_{j=a_2}\delta_{i \in \llbracket j, a_1 \rrbracket}$ is non-negative, we get

$$\deg(f) \geq \min \left(\ell_{(P_i, P_j), (P_{i+1}, P_j)}^\theta + \ell_{(P_{i+1}, P_j), a}^\theta, \ell_{(P_i, P_j), (P_i, P_{j+1})}^\theta + \ell_{(P_i, P_{j+1}), a}^\theta \right).$$

Suppose that $a_1 \neq i$ and $a_2 \neq j$. In this case,

$$\ell_{(P_i, P_j), (P_{i+1}, P_j)}^\theta + \ell_{(P_{i+1}, P_j), a}^\theta = \ell_{(P_i, P_j), (P_i, P_{j+1})}^\theta + \ell_{(P_i, P_{j+1}), a}^\theta.$$

Notice that if $(P_i, P_j) \vdash a$, i.e. $j \in \llbracket a_2, i \rrbracket$ or $a_1 \in \llbracket a_2, i \rrbracket$ then we have $j \in \llbracket a_2, i+1 \rrbracket$ or $a_1 \in \llbracket a_2, i+1 \rrbracket$ or $j+1 \in \llbracket a_2, i \rrbracket$ or $a_1 \in \llbracket a_2, i \rrbracket$, i.e. $(P_{i+1}, P_j) \vdash a$ or $(P_i, P_{j+1}) \vdash a$. From that fact and easy observations, we get that

$$f \in \text{Hom}_{\Lambda'}(M_{(P_{i+1}, P_j)} \oplus M_{(P_i, P_{j+1})}, M_a) u.$$

Suppose that $a_1 = i$. Then $\deg(f) \geq \ell_{(P_i, P_j), (P_i, P_{j+1})}^\theta + \ell_{(P_i, P_{j+1}), a}^\theta$.

If $(P_i, P_j) \vdash a$, i.e. $j \in \llbracket a_2, i \rrbracket$ then we have $j+1 \in \llbracket a_2, i \rrbracket$ or $j+1 = i$, i.e. $(P_i, P_{j+1}) \vdash a$ or $(P_i, P_{j+1}) = (P_i, *) \oplus (P_i, \bowtie)$. From that fact and easy observations, we get that

$$f \in \text{Hom}_{\Lambda'}(M_{(P_i, P_{j+1})}, M_a) u.$$

Suppose that $a_2 = j$. Then $\deg(f) \geq \ell_{(P_i, P_j), (P_{i+1}, P_j)}^\theta + \ell_{(P_{i+1}, P_j), a}^\theta$.

If $(P_i, P_j) \vdash a$, i.e. $a_1 \in \llbracket a_2, i \rrbracket$ then we have $a_1 \in \llbracket a_2, i+1 \rrbracket$, i.e. $(P_{i+1}, P_j) \vdash a$. From that fact and easy observations, we get that

$$f \in \text{Hom}_{\Lambda'}(M_{(P_{i+1}, P_j)}, M_a) u.$$

We finished to prove that in the first case, we have an almost split sequence.

(b) Let us consider the second case (the third case is similar to the second case). Let $f : M_{(P_i,*)} \rightarrow M_a$ be a morphism which is not a split monomorphism. As before, $\deg(f) \geq \ell_{(P_i,*)}^\theta + 2n\delta_{a=(P_i,*)}$.

Notice that, thanks to the beginning of the proof of Lemma 2.15,

$$\begin{aligned} & \ell_{(P_i,*)}^\theta + \ell_{(P_{i+1},P_i)}^\theta - \ell_{(P_i,*)}^\theta \\ &= n(\delta_{i=a_1} + 0 + 0 + |\delta_{i=a_1} - \delta_{a_2 \in \llbracket a_1, i \rrbracket}| - \delta_{a_2 \in \llbracket a_1, i \rrbracket}) \\ &= 2n\delta_{i=a_1} \delta_{a_2 \in \llbracket i, a_1 \rrbracket} = 2n\delta_{a=(P_i,*)} \end{aligned}$$

so $f \in (\text{Hom}_{\Lambda'}(M_{(P_{i+1},P_i)}, M_a)v$. \square

Proposition 3.13. *Denoting $v' = u - v$, the non-split extensions between indecomposable objects of $\text{CM } \Lambda'$ are, up to isomorphism,*

$$\begin{aligned} & 0 \rightarrow M_{(P_i,P_j)} \xrightarrow{\begin{bmatrix} u^{d(j,l)} \\ u^{d(i,k)} \end{bmatrix}} M_{(P_i,P_l)} \oplus M_{(P_k,P_j)} \xrightarrow{\begin{bmatrix} u^{d(i,k)} & -u^{d(j,l)} \end{bmatrix}} M_{(P_k,P_l)} \rightarrow 0 \\ & \text{if } k \in \llbracket i, j \rrbracket \text{ and } l \in \llbracket j, i \rrbracket; \\ & 0 \rightarrow M_{(P_i,P_j)} \xrightarrow{\begin{bmatrix} u^{d(j,k)+n} \\ u^{d(i,l)} \end{bmatrix}} M_{(P_k,P_i)} \oplus M_{(P_l,P_j)} \xrightarrow{\begin{bmatrix} u^{d(i,l)} & -u^{d(j,k)+n} \end{bmatrix}} M_{(P_k,P_l)} \rightarrow 0 \\ & \text{if } k \in \llbracket j, i \rrbracket \text{ and } l \in \llbracket i, j \rrbracket; \\ & 0 \rightarrow M_{(P_i,P_j)} \xrightarrow{\begin{bmatrix} u^{d(j,k)} \\ u^{d(i,l)} \end{bmatrix}} M_{(P_k,P_i)} \oplus M_{(P_l,P_j)} \xrightarrow{\begin{bmatrix} u^{d(i,l)} & -u^{d(j,k)} \end{bmatrix}} M_{(P_k,P_l)} \rightarrow 0 \\ & 0 \rightarrow M_{(P_i,P_j)} \xrightarrow{\begin{bmatrix} u^{d(j,l)} \\ u^{d(i,k)} \end{bmatrix}} M_{(P_l,P_i)} \oplus M_{(P_k,P_j)} \xrightarrow{\begin{bmatrix} u^{d(i,k)} & -u^{d(j,l)} \end{bmatrix}} M_{(P_k,P_l)} \rightarrow 0 \\ & \text{if } l \in \llbracket i, k \rrbracket \text{ and } j \in \llbracket k, i \rrbracket; \\ & 0 \rightarrow M_{(P_i,P_j)} \xrightarrow{\begin{bmatrix} u^{d(i,k)} \\ v^{d(j,i)} \end{bmatrix}} M_{(P_k,P_j)} \oplus M_{(P_i,*)} \xrightarrow{\begin{bmatrix} v^{d(j,k)} & -v^{2d(i,k)} \end{bmatrix}} M_{(P_k,*)} \rightarrow 0 \\ & 0 \rightarrow M_{(P_i,P_j)} \xrightarrow{\begin{bmatrix} u^{d(i,k)} \\ v'^{d(j,i)} \end{bmatrix}} M_{(P_k,P_j)} \oplus M_{(P_i,\bowtie)} \xrightarrow{\begin{bmatrix} v'^{d(j,k)} & -v'^{2d(i,k)} \end{bmatrix}} M_{(P_k,\bowtie)} \rightarrow 0 \\ & \text{if } k \in \llbracket i, j \rrbracket; \\ & 0 \rightarrow M_{(P_i,*)} \xrightarrow{\begin{bmatrix} v^{d(i,k)} \\ v^{2d(i,l)} \end{bmatrix}} M_{(P_k,P_i)} \oplus M_{(P_l,*)} \xrightarrow{\begin{bmatrix} u^{d(i,l)} & -v^{d(l,k)} \end{bmatrix}} M_{(P_k,P_l)} \rightarrow 0 \\ & 0 \rightarrow M_{(P_i,\bowtie)} \xrightarrow{\begin{bmatrix} v'^{d(i,k)} \\ v'^{2d(i,l)} \end{bmatrix}} M_{(P_k,P_i)} \oplus M_{(P_l,\bowtie)} \xrightarrow{\begin{bmatrix} u^{d(i,l)} & -v'^{d(l,k)} \end{bmatrix}} M_{(P_k,P_l)} \rightarrow 0 \\ & \text{if } i \in \llbracket k, l \rrbracket; \\ & 0 \rightarrow M_{(P_i,*)} \xrightarrow{v^{d(i,k)}} M_{(P_k,P_i)} \xrightarrow{v'^{d(i,k)}} M_{(P_k,\bowtie)} \rightarrow 0 \\ & 0 \rightarrow M_{(P_i,\bowtie)} \xrightarrow{v'^{d(i,k)}} M_{(P_k,P_i)} \xrightarrow{v^{d(i,k)}} M_{(P_k,*)} \rightarrow 0 \\ & \text{if } i \neq k. \end{aligned}$$

if $i \neq k$.

Moreover, in each case, fixing representatives of these isomorphism classes of short exact sequences induces a basis of the corresponding extension group.

Proof. First of all, this is easy to check that all these non-split extensions exist (to prove the exactness, the easiest is to project the sequence at each idempotent) and they are non-split and not isomorphic to each other (and therefore linearly independent). Let $i, j, k, l \in \llbracket 1, n \rrbracket$ such that $j \neq i, i+1$ and $l \neq k, k+1$. Thanks to Proposition 3.12, we know that $\text{CM } \Lambda'$ admits an Auslander-Reiten duality

$$\text{Ext}_{\Lambda'}^1(X, Y) \cong \text{Hom}_K(\underline{\text{Hom}}_{\Lambda'}(Y, \tau(X)), K).$$

Then, using Proposition 3.11, we get

$$\begin{aligned} \dim \text{Ext}_{\Lambda'}^1((M_{(P_k, P_l)}, M_{(P_i, P_j)})) &= \dim \underline{\text{Hom}}_{\Lambda'}(M_{(P_i, P_j)}, M_{(P_{k-1}, P_{l-1})}) \\ &= \delta_{i-1 \in \llbracket k-1, l-1 \rrbracket} [\delta_{l \in \llbracket i, j \rrbracket}] + \delta_{j-1 \in \llbracket k-1, l-1 \rrbracket} [\delta_{k \in \llbracket i, j \rrbracket}] \\ &= \delta_{i \in \llbracket k, l \rrbracket} [\delta_{l \in \llbracket i, j \rrbracket}] + \delta_{j \in \llbracket k, l \rrbracket} [\delta_{k \in \llbracket i, j \rrbracket}] \\ &= \begin{cases} 2, & \text{if } l \in \llbracket i, k \rrbracket \text{ and } j \in \llbracket k, i \rrbracket, \\ 1, & \text{if } k \in \llbracket i, j \rrbracket \text{ and } l \in \llbracket j, i \rrbracket, \\ & \text{or } k \in \llbracket j, i \rrbracket \text{ and } l \in \llbracket i, j \rrbracket, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

We also get

$$\begin{aligned} \dim \text{Ext}_{\Lambda'}^1(M_{(P_k, *)}, M_{(P_i, P_j)}) &= \dim \underline{\text{Hom}}_{\Lambda'}(M_{(P_i, P_j)}, M_{(P_{k-1}, \bowtie)}) \\ &= \delta_{i-1 \in \llbracket k-1, k-1 \rrbracket} [\delta_{k \in \llbracket i, j \rrbracket}] = \delta_{k \in \llbracket i, j \rrbracket}; \\ \dim \text{Ext}_{\Lambda'}^1(M_{(P_k, P_l)}, M_{(P_i, *)}) &= \dim \underline{\text{Hom}}_{\Lambda'}(M_{(P_i, *)}, M_{(P_{k-1}, P_{l-1})}) \\ &= \delta_{i-1 \in \llbracket k-1, l-1 \rrbracket} [\delta_{l \in \llbracket i, i \rrbracket}] = \delta_{i \in \llbracket k, l \rrbracket}; \\ \dim \text{Ext}_{\Lambda'}^1(M_{(P_k, *)}, M_{(P_i, *)}) &= \dim \underline{\text{Hom}}_{\Lambda'}(M_{(P_i, *)}, M_{(P_{k-1}, \bowtie)}) = 0; \\ \dim \text{Ext}_{\Lambda'}^1(M_{(P_k, *)}, M_{(P_i, \bowtie)}) &= \dim \underline{\text{Hom}}_{\Lambda'}(M_{(P_i, \bowtie)}, M_{(P_{k-1}, \bowtie)}) \\ &= \delta_{i-1 \in \llbracket k-1, k-1 \rrbracket} [\delta_{k \in \llbracket i, i \rrbracket}] = \delta_{i \neq k}. \end{aligned}$$

The other cases are realized by swapping $*$ and \bowtie . In any case, we exhausted the dimensions with the provided short exact sequences. \square

Corollary 3.14. *If a and b are two tagged arcs of P^* , $\dim \text{Ext}_{\Lambda'}^1(M_a, M_b)$ is the minimal number of intersection points between representatives of their isotopy classes (where $(P_i, *)$ and (P_j, \bowtie) are intersecting once for $i \neq j$ by convention).*

Proof. It is an easy case by case study. \square

3.3. Cluster tilting objects of $\text{CM } \Lambda$ and relation to the cluster category.

Let us recall the definition of cluster tilting objects.

Definition 3.15. Let \mathcal{C} be a triangulated or exact category. An object T in \mathcal{C} is said to be *cluster tilting* if

$$\text{add } T = \{Z \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^1(T, Z) = 0\} = \{Z \in \mathcal{C} \mid \text{Ext}_{\mathcal{C}}^1(Z, T) = 0\},$$

where $\text{add } T$ is the set of finite direct sums of direct summands of T .

For any tagged triangulation σ of the once-punctured polygon P^* , we denote $T_\sigma = \bigoplus_{a \in \sigma} M_a \cong e_F \Gamma_\sigma$.

Theorem 3.16. *The map $\sigma \rightarrow T_\sigma$ gives a one-to-one correspondence between the set of tagged triangulations of P^* and the set of isomorphism classes of basic cluster tilting objects in $\text{CM } \Lambda$.*

Moreover, for any tagged triangulation σ , $\text{End}_\Lambda(T_\sigma) \cong \Gamma_\sigma^{\text{op}}$ through right multiplication.

Proof. Let E be a set of tagged arcs and sides of P^* and $M_E = \bigoplus_{a \in E} M_a$ the corresponding object in $\text{CM } \Lambda$. By Corollary 3.14, any two arcs in E are compatible if and only if $\text{Ext}_\Lambda^1(M_E, M_E) = 0$. Thus, M_E is cluster tilting if and only if it is a maximal set of compatible tagged arcs and sides of P^* if and only if $E = \sigma$ is a tagged triangulation of P^* . Thus, $M_E = T_\sigma$.

For the second part, thanks to Propositions 2.26 and 3.10, for any $a, b \in \sigma$, $\text{Hom}_{\Lambda'}(M_a, M_b) \cong A_{a,b} \cong e_a \Gamma_\sigma e_b$. Therefore,

$$\text{End}_{\Lambda'}(T_\sigma) = \bigoplus_{a,b \in \sigma} e_a \Gamma_\sigma e_b = \Gamma_\sigma.$$

Moreover, the composition on the left coincides with the multiplication on the right by Propositions 2.26 and 3.10. Notice that we get the opposite algebra because we make endomorphism rings act on the left. \square

Theorem 3.3 and 3.16 show that the category $\text{CM } \Lambda$ is very similar to the cluster category of type D_n . In the rest of this section, we give an explicit connection. First, we recall some basic facts about cluster categories. The cluster category is defined in [6] as follows.

Definition 3.17. For an acyclic quiver Q , the *cluster category* $\mathcal{C}(KQ)$ is the orbit category $\mathcal{D}^b(KQ)/F$ of the bounded derived category $\mathcal{D}^b(KQ)$ by the functor $F = \tau^{-1}[1]$, where τ denotes the Auslander-Reiten translation and $[1]$ denotes the shift functor. The objects in $\mathcal{C}(KQ)$ are the same as in $\mathcal{D}^b(KQ)$, and the morphisms are given by

$$\text{Hom}_{\mathcal{C}(KQ)}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(KQ)}(F^i X, Y),$$

where X and Y are objects in $\mathcal{D}^b(KQ)$. For $f \in \text{Hom}_{\mathcal{C}(KQ)}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}(KQ)}(Y, Z)$, the composition is defined by

$$(g \circ f)_i = \sum_{i_1 + i_2 = i} g_{i_1} \circ F^{i_1}(f_{i_2})$$

for all $i \in \mathbb{Z}$.

In [15], Happel proved that $\mathcal{D}^b(KQ)$ has Auslander-Reiten triangles. For a Dynkin quiver Q , he shows in [14] that the Auslander-Reiten quiver of $\mathcal{D}^b(KQ)$ is $\mathbb{Z}\Delta$, where Δ is the underlying Dynkin diagram of Q . Then the Auslander-Reiten quiver of $\mathcal{C}(KQ)$ is $\mathbb{Z}\Delta/\varphi$, where φ is the graph automorphism induced by $\tau^{-1}[1]$. In type D_n , the Auslander-Reiten quiver of \mathcal{C} has the shape of a cylinder with n τ -orbits. As a quiver, it is the same as the quiver of $\underline{\text{CM}} \Lambda$ (see Figures 3.1, 3.2).

Recall that a triangulated category is said to be algebraic if it is the stable category of a Frobenius category. Let us state the following result by Keller and Reiten.

Theorem 3.18 ([24, Introduction and Appendix]). *If K is a perfect field and \mathcal{C} an algebraic 2-Calabi-Yau triangulated category containing a cluster tilting object T with $\text{End}_{\mathcal{C}}(T) \cong KQ$ hereditary, then there is a triangle-equivalence $\mathcal{C}(KQ) \rightarrow \mathcal{C}$.*

By using the above statements, we show the following triangle-equivalences between cluster categories of type D and stable categories of Cohen-Macaulay modules.

Theorem 3.19. (1) *The stable category $\underline{\text{CM}} \Lambda$ is 2-Calabi-Yau.*
 (2) *If K is perfect, then there is a triangle-equivalence $\mathcal{C}(KQ) \cong \underline{\text{CM}} \Lambda$ for a quiver Q of type D_n .*

Proof. We will prove (1) in the next subsection independently.

Let σ be the triangulation of P^* whose set of tagged arcs is

$$\{(P_1, P_3), (P_1, P_4), \dots, (P_1, P_n), (P_1, *), (P_1, \bowtie)\}.$$

The full subquiver Q of Q_σ with set of vertices $Q_{\sigma,0} \setminus F$ is a quiver of type D_n . Thus, we have

$$\Gamma_\sigma^{\text{op}}/(e_F) \cong (KQ)^{\text{op}}.$$

By Theorem 3.16, for the cluster tilting object T_σ , we have the following isomorphism

$$\underline{\text{End}}_\Lambda(T_\sigma) \cong \Gamma_\sigma^{\text{op}}/(e_F).$$

Then, by Theorem 3.18, we have $\mathcal{C}((KQ)^{\text{op}}) \cong \underline{\text{CM}} \Lambda$. \square

3.4. Proof of Theorem 3.19 (1). In this subsection, we prove that the stable category $\underline{\text{CM}} \Lambda$ is 2-Calabi-Yau. Throughout, we denote $D_K := \text{Hom}_K(-, K)$, $D_R := \text{Hom}_R(-, R)$ and $(-)^* := \text{Hom}_\Lambda(-, \Lambda)$.

Let us recall some general definitions and facts about Cohen-Macaulay modules. Let A be an R -order.

Definition 3.20. We say that X is an *injective* Cohen-Macaulay A -module if $\text{Ext}_A^1(Y, X) = 0$ for any $Y \in \text{CM } A$, or equivalently, $X \in \text{add}(\text{Hom}_R(A^{\text{op}}, R))$. Denote by $\text{inj } A$ the category of *injective* Cohen-Macaulay A -modules.

An R -order A is *Gorenstein* if $\text{Hom}_R(A_A, R)$ is projective as a left A -module, or equivalently, if $\text{Hom}_R({}_A A, R)$ is projective as a right A -module. We have an exact duality $D_R : \text{CM } A^{\text{op}} \rightarrow \text{CM } A$.

The Nakayama functor is defined here by $\nu : \text{proj } A \xrightarrow{(-)^*} \text{proj } A^{\text{op}} \xrightarrow{D_R} \text{inj } A$, which is isomorphic to $(D_R A) \otimes_A -$. For any Cohen-Macaulay A -module X , consider a projective presentation

$$P_1 \xrightarrow{f} P_0 \rightarrow X \rightarrow 0$$

and apply $(-)^* : \text{mod } A \rightarrow \text{mod } A^{\text{op}}$ to it to get the following exact sequence:

$$0 \rightarrow X^* \rightarrow P_0^* \xrightarrow{f^*} P_1^* \rightarrow \text{coker}(f^*) \rightarrow 0.$$

We denote $\text{coker}(f^*)$ by $\text{Tr } X$ and we get $\text{Im}(f^*) = \Omega \text{Tr } X$, where Ω is the syzygy functor: $\underline{\text{mod}} A^{\text{op}} \rightarrow \underline{\text{mod}} A^{\text{op}}$. Then we apply $D_R : \text{CM } A^{\text{op}} \rightarrow \text{CM } A$ to

$$0 \rightarrow X^* \rightarrow P_0^* \xrightarrow{f^*} \Omega \text{Tr } X \rightarrow 0$$

and denote $\tau X := D_R \Omega \text{Tr } X$, we get the exact sequence

$$0 \rightarrow \tau X \rightarrow \nu P_0 \rightarrow \nu X \rightarrow 0. \quad (3.21)$$

For an R -order A , if $K(x) \otimes_R A$ is a semisimple $K(x)$ -algebra, then we call A an isolated singularity. By using the notions above, we have the following well-known results in Auslander-Reiten theory.

Theorem 3.22. [3, 28, 29] *Let A be an R -order. If A is an isolated singularity, then*

- (1) [3, Chapter I, Proposition 8.3] *The construction τ gives an equivalence $\underline{\text{CM}} A \rightarrow \overline{\text{CM}} A$, where $\overline{\text{CM}} A$ is the quotient of $\text{CM } A$ with respect to the subgroup of maps which factor through an injective object.*
- (2) [3, Chapter I, Proposition 8.7] *For $X, Y \in \underline{\text{CM}} A$, there is a functorial isomorphism*

$$\underline{\text{Hom}}_A(X, Y) \cong D_K \text{Ext}_A^1(Y, \tau X).$$

For Gorenstein orders, we have the following nice properties.

Proposition 3.23. *Assume that A is a Gorenstein isolated singularity, then we have*

- (1) $\text{CM } A$ is a Frobenius category.
- (2) $\underline{\text{CM}} A$ is a K -linear Hom-finite triangulated category.
- (3) $\tau = \Omega\nu = [-1] \circ \nu$.

Proof. (1) The projective objects in $\text{CM } A$ are just projective A -modules. They are also injective objects. Since each finitely generated A -module is a quotient of a projective A -module, it follows that $\text{CM } A$ is a Frobenius category; (2) is due to [14] and [31, Lemma 3.3]; (3) is a direct consequence of (3.21). \square

The order Λ is Gorenstein. Indeed, as a graded left Λ -module,

$$\text{D}_R(\Lambda_\Lambda) = \text{Hom}_R \left(\begin{bmatrix} R' & R' & R' & \cdots & R' & R' & X^{-1}(X, Y) \\ (X, Y) & R' & R' & \cdots & R' & R' & R' \\ (X) & (X, Y) & R' & \cdots & R' & R' & R' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (X) & (X) & (X) & \cdots & R' & R' & R' \\ (X) & (X) & (X) & \cdots & (X, Y) & R' & R' \\ (X) & (X) & (X) & \cdots & (X) & (X, Y) & R' \end{bmatrix}, R \right)$$

can be identified with

$$\begin{aligned} & X^{-1} \begin{bmatrix} R' & X^{-1}(X, Y) & X^{-1}R' & \cdots & X^{-1}R' & X^{-1}R' & X^{-1}R' \\ R' & R' & X^{-1}(X, Y) & \cdots & X^{-1}R' & X^{-1}R' & X^{-1}R' \\ R' & R' & R' & \cdots & X^{-1}R' & X^{-1}R' & X^{-1}R' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ R' & R' & R' & \cdots & R' & X^{-1}(X, Y) & X^{-1}R' \\ R' & R' & R' & \cdots & R' & R' & X^{-1}(X, Y) \\ (X, Y) & R' & R' & \cdots & R' & R' & R' \end{bmatrix} \\ &= \Lambda V^{-1} \subset \text{M}_n(R'[X^{-1}]), \end{aligned}$$

where

$$V = \begin{bmatrix} 0 & 0 & \cdots & 0 & X & 0 \\ 0 & 0 & \cdots & 0 & 0 & X \\ X^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & X^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & X^2 & 0 & 0 \end{bmatrix}.$$

Therefore $\text{D}_R(\Lambda_\Lambda)$ is a projective (left) Λ -module.

According to Theorem 3.22 and Proposition 3.23, we have

$$\underline{\text{Hom}}_\Lambda(X, Y) \cong \text{D}_K \underline{\text{Hom}}_\Lambda(Y, \nu X)$$

for $X, Y \in \text{CM } \Lambda$. Thus $\nu = (\text{D}_R \Lambda) \otimes_\Lambda -$ is a Serre functor. We want to prove that

$$(\text{D}_R \Lambda) \otimes_\Lambda - \cong \Omega^{-2}(-).$$

Thanks to the previous discussion, there is an isomorphism of Λ -modules:

$$f : \Lambda \rightarrow \text{D}_R(\Lambda_\Lambda), \mu \mapsto \mu V^{-1}.$$

We define the automorphism α of Λ by $\alpha(\lambda) = V^{-1}\lambda V$ for $\lambda \in \Lambda$. The automorphism α corresponds to a $4\pi/n$ counter-clockwise rotation of the quiver of Λ

shown page 25. In fact, if

$$\lambda = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n-2} & \lambda_{1,n-1} & \lambda_{1,n} \\ \lambda_{2,1} & \lambda_{2,2} & \cdots & \lambda_{2,n-2} & \lambda_{2,n-1} & \lambda_{2,n} \\ \lambda_{3,1} & \lambda_{3,2} & \cdots & \lambda_{3,n-2} & \lambda_{3,n-1} & \lambda_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{n-2,1} & \lambda_{1,2} & \cdots & \lambda_{n-2,n-2} & \lambda_{n-2,n-1} & \lambda_{n-2,n} \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{n-1,n-2} & \lambda_{n-1,n-1} & \lambda_{n-1,n} \\ \lambda_{n,1} & \lambda_{n,2} & \cdots & \lambda_{n,n-2} & \lambda_{n,n-1} & \lambda_{n,n} \end{bmatrix}$$

is an element in Λ , then

$$\alpha(\lambda) = \begin{bmatrix} \lambda_{3,3} & \lambda_{3,4} & \cdots & \lambda_{3,n} & X^{-1}\lambda_{3,1} & X^{-1}\lambda_{3,2} \\ \lambda_{4,3} & \lambda_{4,4} & \cdots & \lambda_{4,n} & X^{-1}\lambda_{4,1} & X^{-1}\lambda_{4,2} \\ \lambda_{5,3} & \lambda_{5,4} & \cdots & \lambda_{5,n} & X^{-1}\lambda_{5,1} & X^{-1}\lambda_{5,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{n,3} & \lambda_{n,4} & \cdots & \lambda_{n,n} & X^{-1}\lambda_{n,1} & X^{-1}\lambda_{n,2} \\ X\lambda_{1,3} & X\lambda_{1,4} & \cdots & X\lambda_{1,n} & \lambda_{1,1} & \lambda_{1,2} \\ X\lambda_{2,3} & X\lambda_{2,4} & \cdots & X\lambda_{2,n} & \lambda_{2,1} & \lambda_{2,2} \end{bmatrix}.$$

Let A and B be two R -orders. We define ${}_{\vartheta}M_{\varsigma}$ for an (A, B) -bimodule M , $\vartheta \in \text{Aut}(A)$ and $\varsigma \in \text{Aut}(B)$ as follows: ${}_{\vartheta}M_{\varsigma} := M$ as a vector space and the (A, B) -bimodule structure is given by

$$a \times m \times b = \vartheta(a)m\varsigma(b)$$

for $m \in {}_{\vartheta}M_{\varsigma}$ and $a \in A$, $b \in B$. Since $\vartheta \in \text{Aut}(A)$, $\vartheta(-)$ is an automorphism of $\text{mod } A$.

Proposition 3.24. *The above $f : \Lambda \rightarrow D_R \Lambda$ gives an isomorphism of Λ -bimodules*

$${}_1\Lambda_{\alpha} \cong D_R \Lambda.$$

Proof. Clearly, f preserves the left action of Λ . Moreover, it preserves the right action since for $\lambda, \mu \in \Lambda$, we have

$$f(\mu\alpha(\lambda)) = f(\mu(V^{-1}\lambda V)) = \mu(V^{-1}\lambda V)V^{-1} = \mu V^{-1}\lambda = f(\mu)\lambda. \quad \square$$

By using the isomorphism of Proposition 3.24, we find the following description of the Nakayama functor ν .

Lemma 3.25. *We have an isomorphism $\nu \cong {}_{\alpha^{-1}}(-)$ of endofunctors of $\underline{\text{CM}} \Lambda$.*

Proof. Since $D_R \Lambda \cong {}_1\Lambda_{\alpha}$, it follows that $\nu \cong {}_1\Lambda_{\alpha} \otimes_{\Lambda} -$. On the other hand, we have an isomorphism $H : {}_1\Lambda_{\alpha} \otimes_{\Lambda} - \cong {}_{\alpha^{-1}}(-)$ given by $\lambda \otimes - \mapsto \alpha^{-1}(\lambda)(-)$. Thus the assertion follows. \square

Let $T = K[x, y]$ and $S := K[x, y]/(p)$ for some $p \in T$.

We define a $\mathbb{Z}/n\mathbb{Z}$ -grading on T by setting $\deg(x) = 1$ and $\deg(y) = -1$. This makes T a $\mathbb{Z}/n\mathbb{Z}$ -graded algebra

$$T = \bigoplus_{\bar{i} \in \mathbb{Z}/n\mathbb{Z}} T_{\bar{i}} = T_{\bar{0}} \oplus T_{\bar{1}} \oplus \cdots \oplus T_{\overline{n-1}}.$$

Suppose that p is homogeneous of degree d with respect to this grading. Then the quotient ring

$$S = K[x, y]/(p) = S_{\bar{0}} \oplus S_{\bar{1}} \oplus \cdots \oplus S_{\overline{n-1}}$$

has a natural structure of a $\mathbb{Z}/n\mathbb{Z}$ -graded algebra. The following result can be easily established from classical results about matrix factorization (see [10, Theorem 3.22] for a detailed proof).

Theorem 3.26 ([31]). *In the category $\underline{\text{CM}}^{\mathbb{Z}/n\mathbb{Z}}(S)$, there is an isomorphism of autoequivalences $[2] \cong (-d)$.*

Setting $p := x^{n-1}y - y^2$, we have $S = K[x, y]/(x^{n-1}y - y^2)$. Identifying $R' = K[X, Y]/(XY - Y^2)$ as a subalgebra of S via $X \mapsto x^n$ and $Y \mapsto xy$, we regard S as an R' -algebra. We obtain the following lemma.

Lemma 3.27. *For $i \in \llbracket 0, n-1 \rrbracket$, as R' -modules, we have*

$$S_{\bar{i}} \cong \begin{cases} R'x^i, & \text{if } i \in \llbracket 0, n-2 \rrbracket, \\ (1, X^{-1}Y)x^{n-1}, & \text{if } i = n-1. \end{cases}$$

Proof. Let $i \in \llbracket 0, n-2 \rrbracket$. Over $R = K[X]$, $S_{\bar{i}}$ is generated by x^i and $x^{i+1}y$. Thus, we have $S_{\bar{i}} \cong R'x^i$. Over R , $S_{\overline{n-1}}$ is generated by x^{n-1} and y . So $S_{\overline{n-1}} \cong (1, X^{-1}Y)x^{n-1}$. \square

We define an R -order $S^{[n]}$ which is a subalgebra of $M_n(S)$ from the $\mathbb{Z}/n\mathbb{Z}$ -graded algebra S as follows:

$$S^{[n]} = \begin{bmatrix} S_{\bar{0}} & S_{\bar{1}} & S_{\bar{2}} & \cdots & S_{\overline{n-2}} & S_{\overline{n-1}} \\ S_{\overline{n-1}} & S_{\bar{0}} & S_{\bar{1}} & \cdots & S_{\overline{n-3}} & S_{\overline{n-2}} \\ S_{\overline{n-2}} & S_{\overline{n-1}} & S_{\bar{0}} & \cdots & S_{\overline{n-4}} & S_{\overline{n-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{\bar{2}} & S_{\bar{3}} & S_{\bar{4}} & \cdots & S_{\bar{0}} & S_{\bar{1}} \\ S_{\bar{1}} & S_{\bar{2}} & S_{\bar{3}} & \cdots & S_{\overline{n-1}} & S_{\bar{0}} \end{bmatrix}.$$

Proposition 3.28. *We have an isomorphism $S^{[n]} \cong \Lambda$ of R' -algebras.*

Proof. According to Lemma 3.27, we have

$$S^{[n]} = \begin{bmatrix} R' & R'x & R'x^2 & \cdots & R'x^{n-2} & (1, X^{-1}Y)x^{n-1} \\ (1, X^{-1}Y)x^{n-1} & R' & R'x & \cdots & R'x^{n-3} & R'x^{n-2} \\ R'x^{n-2} & (1, X^{-1}Y)x^{n-1} & R' & \cdots & R'x^{n-4} & R'x^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R'x^2 & R'x^3 & R'x^4 & \cdots & R' & R'x \\ R'x & R'x^2 & R'x^3 & \cdots & (1, X^{-1}Y)x^{n-1} & R' \end{bmatrix}.$$

Taking the conjugation by the diagonal matrix $B = \text{diag}(x^i)_{i \in \llbracket 0, n \rrbracket}$, we get

$$BS^{[n]}B^{-1} = \Lambda.$$

\square

From now on, we identify Λ and $S^{[n]}$. Consider the matrix

$$U = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}.$$

The automorphism β of $S^{[n]}$ which is given by $\beta(s) = U^{-1}sU$ for $s \in S^{[n]}$ corresponds to the automorphism α of Λ . Thus we have an isomorphism ${}_1S_{\beta^{-1}}^{[n]} \cong {}_1\Lambda_{\alpha^{-1}}$ of $S^{[n]}$ -bimodules.

Using the same notation above, we have the following lemma.

Lemma 3.29. (1) [19, Theorem 3.1] *The functor*

$$F : \text{mod } \mathbb{Z}/n\mathbb{Z}S \rightarrow \text{mod } S^{[n]}$$

$$M_{\bar{0}} \oplus M_{\bar{1}} \oplus \dots \oplus M_{\overline{n-1}} \mapsto [M_{\bar{0}} \ M_{\bar{1}} \ \dots \ M_{\overline{n-1}}]^t$$

is an equivalence of categories.

(2) For $i \in \mathbb{Z}$, we denote by $(i) : \text{mod } \mathbb{Z}/n\mathbb{Z}S \rightarrow \text{mod } \mathbb{Z}/n\mathbb{Z}S$ the grade shift functor defined by $M(i)_{\bar{j}} := M_{\overline{i+j}}$ for $M \in \text{mod } \mathbb{Z}/n\mathbb{Z}S$. The grade shift functor (i) induces an autofunctor (denoted by γ_i) in $\text{mod } S^{[n]}$ which makes the following diagram commute:

$$\begin{array}{ccc} \text{mod } \mathbb{Z}/n\mathbb{Z}S & \xrightarrow{F} & \text{mod } S^{[n]} \\ (i) \downarrow & & \downarrow \gamma_i \\ \text{mod } \mathbb{Z}/n\mathbb{Z}S & \xrightarrow{F} & \text{mod } S^{[n]}. \end{array}$$

More precisely, for any left $S^{[n]}$ -module $[M_{\bar{0}} \ M_{\bar{1}} \ \dots \ M_{\overline{n-1}}]^t$, we have

$$\gamma_i \left([M_{\bar{0}} \ M_{\bar{1}} \ \dots \ M_{\overline{n-1}}]^t \right) = [M_{\bar{i}} \ M_{\overline{i+1}} \ \dots \ M_{\overline{i+n-1}}]^t.$$

Now we can prove the 2-Calabi-Yau property of $\underline{\text{CM}} \Lambda$.

Proof of Theorem 3.19 (1). The equivalence $\text{mod } \mathbb{Z}/n\mathbb{Z}S \cong \text{mod } S^{[n]} = \text{mod } \Lambda$ induces an equivalence

$$\text{CM}^{\mathbb{Z}/n\mathbb{Z}}(S) \cong \text{CM } S^{[n]} = \text{CM } \Lambda.$$

In the category $\underline{\text{CM}}^{\mathbb{Z}/n\mathbb{Z}} S$, according to Theorem 3.26, we have an isomorphism of functors

$$[2] \cong (-\deg(x^{n-2} - y^2)) = (2).$$

By Lemma 3.25, we have $\nu \cong \alpha_{-1}(-)$. Therefore, it is enough to prove $\alpha_{-1}(-) \cong (2)$. This is equivalent to prove that ${}_{\beta}M \cong \gamma_{-2}(M)$ holds for any $M \in \text{CM } S^{[n]}$.

Let s_i be the row matrix which has 1 in the i -th column. Since

$$\begin{aligned} {}_{\beta}M &= [s_0 \times {}_{\beta}M \ s_1 \times {}_{\beta}M \ s_2 \times {}_{\beta}M \ \dots \ s_{n-1} \times {}_{\beta}M]^t \\ &= [\beta(s_0)M \ \beta(s_1)M \ \beta(s_2)M \ \dots \ \beta(s_{n-1})M]^t \\ &= [s_{n-2}M \ s_{n-1}M \ s_0M \ \dots \ s_{n-3}M]^t, \end{aligned}$$

it follows that ${}_{\beta}M \cong \gamma_{-2}(M)$. Therefore, the category $\underline{\text{CM}} \Lambda$ is 2-Calabi-Yau. \square

4. GRADED COHEN-MACAULAY Λ -MODULES

In this section, we prove a graded version of Theorem 3.19 which gives a relationship between the category $\text{CM}^{\mathbb{Z}}(\Lambda)$ of graded Cohen-Macaulay Λ -modules and the bounded derived category $\mathcal{D}^b(KQ)$ of type D_n .

Let Q be an acyclic quiver. We denote by $\mathcal{K}^b(\text{proj } KQ)$ the bounded homotopy category of finitely generated projective KQ -modules, and by $\mathcal{D}^b(KQ)$ the bounded derived category of finitely generated KQ -modules. These are triangulated categories and the canonical embedding $\mathcal{K}^b(\text{proj } KQ) \rightarrow \mathcal{D}^b(KQ)$ is a triangle functor.

We define a grading on Λ by $\Lambda_i = \Lambda \cap M_n(KX^i + KX^{i-1}Y)$ for $i \in \mathbb{Z}$. This makes $\Lambda = \bigoplus_{i \in \mathbb{Z}} \Lambda_i$ a \mathbb{Z} -graded algebra. The category of graded Cohen-Macaulay Λ -modules, $\text{CM}^{\mathbb{Z}}(\Lambda)$, is defined as follows. The objects are graded Λ -modules which are Cohen-Macaulay, and the morphisms in $\text{CM}^{\mathbb{Z}}(\Lambda)$ are Λ -morphisms preserving the degree. The category $\text{CM}^{\mathbb{Z}}(\Lambda)$ is a Frobenius category. Its stable category is denoted by $\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)$. For $i \in \mathbb{Z}$, we denote by $(i) : \text{CM}^{\mathbb{Z}}(\Lambda) \rightarrow \text{CM}^{\mathbb{Z}}(\Lambda)$ the grade

shift functor: Given a graded Cohen-Macaulay Λ -module X , we define $X(i)$ to be X as a Λ -module, with the grading $X(i)_j = X_{i+j}$ for any $j \in \mathbb{Z}$.

Remark 4.1. We show that this grading of Λ is analogous to the grading of Λ' given by the θ -length. Let $i, j \in F$. By Theorem 2.30, $e_i \Lambda' e_j \cong e_i \Lambda e_j$ holds. Let $\lambda \in e_i \Lambda' e_j \cong e_i \Lambda e_j$. Using a similar argument as in the proof of Theorem 2.30, we have

$$\frac{\ell^\theta(\lambda) + 2d(1, i) - 2d(1, j)}{2n} = \deg(\lambda)$$

where $\deg(\lambda)$ is the degree of λ as a member of Λ . Consider the two graded algebras

$$\Lambda' = \bigoplus_{i=1}^n \Lambda' e_i \quad \text{and} \quad \Lambda'' := \text{End} \left(\bigoplus_{i=1}^n u^{2d(1, i)} \Lambda' e_i \right).$$

By graded Morita equivalence, we have $\text{CM}^{\mathbb{Z}} \Lambda' \cong \text{CM}^{\mathbb{Z}}(\Lambda'')$. Since $\Lambda \cong \Lambda''$ as R -orders and $\deg(X) = 2n$ in Λ'' , it follows that the Auslander-Reiten quiver of $\text{CM}^{\mathbb{Z}}(\Lambda'')$ has $2n$ connected components each of which is a degree shift of the Auslander-Reiten quiver of $\text{CM}^{\mathbb{Z}} \Lambda$.

We introduce the properties of $\text{CM}^{\mathbb{Z}} \Lambda$ in the following theorems.

Theorem 4.2. (1) *The set of isomorphism classes of indecomposable objects of $\text{CM}^{\mathbb{Z}} \Lambda$ is*

$$\{(i, j) \mid i, j \in \mathbb{Z}, 0 < j - i < n\} \cup \{(i, *) \mid i \in \mathbb{Z}\} \cup \{(i, \bowtie) \mid i \in \mathbb{Z}\},$$

where

$$(i, j) = {}^t \left[\overbrace{(X) \cdots (X)}^i \quad \overbrace{(X^2, Y^2) \cdots (X^2, Y^2)}^{j-i} \quad \overbrace{(X^2) \cdots (X^2)}^{n-j} \right] \quad \text{if } 0 < i < j \leq n;$$

$$(i, j) = {}^t \left[\overbrace{(X, Y) \cdots (X, Y)}^{j-n} \quad \overbrace{(X) \cdots (X)}^{n-j+i} \quad \overbrace{(X^2, Y^2) \cdots (X^2, Y^2)}^{n-i} \right] \quad \text{if } i \leq n < j;$$

$$(i, *) = {}^t \left[\overbrace{(Y) \cdots (Y)}^i \quad \overbrace{(Y^2) \cdots (Y^2)}^{n-i} \right] \quad \text{if } 0 < i \leq n;$$

$$(i, \bowtie) = {}^t \left[\overbrace{(X - Y) \cdots (X - Y)}^i \quad \overbrace{(X^2 - Y^2) \cdots (X^2 - Y^2)}^{n-i} \right] \quad \text{if } 0 < i \leq n,$$

and the other (i, j) are obtained by shift:

$$(i + kn, j + kn) = (i, j)(k)$$

$$(i + kn, *) = (i, *)(k)$$

$$(i + kn, \bowtie) = (i, \bowtie)(k)$$

for $k \in \mathbb{Z}$. The projective-injective objects are of the form $(i, i + 1)$ for $i \in \mathbb{Z}$.

(2) *The non-split extensions of indecomposable objects of $\text{CM}^{\mathbb{Z}} \Lambda$ are of the form*

$$0 \rightarrow (i, j) \rightarrow (i, l) \oplus (k, j) \rightarrow (k, l) \rightarrow 0 \quad \text{if } i < k < j < l < i + n;$$

$$0 \rightarrow (i, j) \rightarrow (k, i + n) \oplus (l, j) \rightarrow (k, l + n) \rightarrow 0 \quad \text{if } i < l < j \leq k < i + n;$$

$$\left. \begin{array}{l} 0 \rightarrow (i, j) \rightarrow (k, i + n) \oplus (l, j) \rightarrow (k, l + n) \rightarrow 0 \\ 0 \rightarrow (i, j) \rightarrow (l, i + n) \oplus (k, j) \rightarrow (k, l + n) \rightarrow 0 \end{array} \right\} \quad \text{if } i < l < k < j < i + n;$$

$$\left. \begin{array}{l} 0 \rightarrow (i, j) \rightarrow (k, j) \oplus (i, *) \rightarrow (k, *) \rightarrow 0 \\ 0 \rightarrow (i, j) \rightarrow (k, j) \oplus (i, \bowtie) \rightarrow (k, \bowtie) \rightarrow 0 \end{array} \right\} \quad \text{if } i < k < j < i + n;$$

$$\left. \begin{array}{l} 0 \rightarrow (i, *) \rightarrow (k, i+n) \oplus (l, *) \rightarrow (k, l+n) \rightarrow 0 \\ 0 \rightarrow (i, \bowtie) \rightarrow (k, i+n) \oplus (l, \bowtie) \rightarrow (k, l+n) \rightarrow 0 \end{array} \right\} \quad \text{if } i < l < k < i+n;$$

$$\left. \begin{array}{l} 0 \rightarrow (i, *) \rightarrow (k, i+n) \rightarrow (k, \bowtie) \rightarrow 0 \\ 0 \rightarrow (i, \bowtie) \rightarrow (k, i+n) \rightarrow (k, *) \rightarrow 0 \end{array} \right\} \quad \text{if } i < k < i+n.$$

Moreover, fixing representatives of these isomorphism classes of short exact sequences induces bases of the corresponding extension groups.

(3) The exact category $\text{CM}^{\mathbb{Z}} \Lambda$ admits the Auslander-Reiten sequences

$$0 \longrightarrow (i, j) \longrightarrow (i, j+1) \oplus (i+1, j) \longrightarrow (i+1, j+1) \longrightarrow 0$$

for $i+1 < j < i+n$ (with the convention that $(i, i+n) = (i, *) \oplus (i, \bowtie)$);

$$0 \longrightarrow (i, *) \longrightarrow (i+1, i+n) \longrightarrow (i+1, \bowtie) \longrightarrow 0$$

$$\text{and } 0 \longrightarrow (i, \bowtie) \longrightarrow (i+1, i+n) \longrightarrow (i+1, *) \longrightarrow 0$$

for any $i \in \mathbb{Z}$.

(4) The Auslander-Reiten quiver of $\text{CM}^{\mathbb{Z}}(\Lambda)$ is the repetitive quiver of type D_{n+1} (unfolded version of Figures 3.1 and 3.2).

(5) The syzygy in $\text{CM}^{\mathbb{Z}}(\Lambda)$ is defined on indecomposable objects by:

$$\Omega((i, j)) = (i+1-n, j+1-n); \quad \Omega((i, *)) = (i+1-n, \bowtie); \quad \Omega((i, \bowtie)) = (i+1-n, *).$$

Proof. (1) First of all, it is immediate that the graded modules (i, j) for $0 < j-i < n$, $(i, *)$ and (i, \bowtie) for $i \in \mathbb{Z}$ are not isomorphic. Therefore, we need to prove that there are no other isomorphism classes. We consider the degree forgetful functor $F : \text{CM}^{\mathbb{Z}} \Lambda \rightarrow \text{CM} \Lambda$. Let $X \in \text{CM}^{\mathbb{Z}} \Lambda$ be indecomposable and M be an indecomposable direct summand of FX in $\text{CM} \Lambda$. Using Theorem 3.3, there exists a tagged arc or a side a of P^* such that $M \cong M_a$. Then, it is immediate that $M \cong FY$ where $Y \in \text{CM}^{\mathbb{Z}} \Lambda$ is (i, j) for $0 < j-i < n$ or $(i, *)$ or (i, \bowtie) for $i \in \mathbb{Z}$. There are two morphisms $f : FY \rightarrow FX$ and $g : FX \rightarrow FY$ such that $gf = \text{Id}_{FY}$. Let us write

$$f = \sum_{m \in \mathbb{Z}} f_m \quad \text{and} \quad g = \sum_{m \in \mathbb{Z}} g_m$$

where f_m is a graded morphism from Y to $X(m)$ and g_m a graded morphism from $X(m)$ to Y . Thus, we have

$$\sum_{k \in \mathbb{Z}} g_k f_k = \text{Id}_{FY}$$

and, as the endomorphism ring of Y is K , there exists $k \in \mathbb{Z}$ such that $g_k f_k$ is a non-zero multiple of Id_Y . In other terms we found two graded morphisms $\tilde{f} : Y \rightarrow X(k)$ and $\tilde{g} : X(k) \rightarrow Y$ such that $\tilde{g}\tilde{f} = \text{Id}_Y$. Thus, in $\text{mod}^{\mathbb{Z}}(\Lambda)$, we have an isomorphism $X \cong Y(-k) \oplus X'$. As $\text{mod}^{\mathbb{Z}}(R)$ is Krull-Schmidt, X' is necessarily a graded Cohen-Macaulay module. Finally, as X is indecomposable in $\text{CM}^{\mathbb{Z}} \Lambda$, we get that $X \cong Y(-k)$.

Thus, the set of isomorphism classes of indecomposable graded Cohen-Macaulay Λ -modules is

$$\{(i, j) \mid i, j \in \mathbb{Z}, 0 < j-i < n\} \cup \{(i, *) \mid i \in \mathbb{Z}\} \cup \{(i, \bowtie) \mid i \in \mathbb{Z}\}.$$

Statements (2) and (3) are direct consequences through F of the ungraded versions of Propositions 3.12 and 3.13. The statement (4) is a direct consequence of (1) and (3).

For (5), using (2), the short exact sequences constructed from projective covers are:

$$0 \rightarrow (i+1-n, j+1-n) \rightarrow (i, i+1) \oplus (j-n, j-n+1) \rightarrow (i, j) \rightarrow 0;$$

$$0 \rightarrow (i+1-n, \bowtie) \rightarrow (i, i+1) \rightarrow (i, *) \rightarrow 0;$$

$$0 \rightarrow (i+1-n, *) \rightarrow (i, i+1) \rightarrow (i, \bowtie) \rightarrow 0. \quad \square$$

For any indecomposable graded Cohen-Macaulay Λ -module A , if A is of the form (i, j) for two integers i and j , we write $A_1 = i$ and $A_2 = j$, and if A is of the form $(i, *)$ or (i, \bowtie) , we write $A_1 = i$ and $A_2 = i + n$. In this way, all morphisms in $\text{CM}^{\mathbb{Z}} \Lambda$ are going in the increasing direction in term of these pairs of integers.

Definition 4.3. Let \mathcal{C} be a triangulated category. An object T is said to be *tilting* if $\text{Hom}_{\mathcal{C}}(T, T[k]) = 0$ for any $k \neq 0$ and $\text{thick}(T) = \mathcal{C}$, where $\text{thick}(T)$ is the smallest full triangulated subcategory of \mathcal{C} containing T and closed under isomorphisms and direct summands.

Theorem 4.4 ([23, Theorem 4.3], [20, Theorem 2.2], [4]). *Let \mathcal{C} be an algebraic triangulated Krull-Schmidt category. If \mathcal{C} has a tilting object T , then there exists a triangle-equivalence*

$$\mathcal{C} \rightarrow \mathcal{K}^b(\text{proj End}_{\mathcal{C}}(T)).$$

Now we get the following theorem which is analogous to Theorem 3.16 and Theorem 3.19 (2).

Theorem 4.5. *Let Q be a quiver of type D_n . Then*

- (1) *for a tagged triangulation σ of the once-punctured polygon P^* , the cluster tilting object $e_F \Gamma_{\sigma}$ can be lifted to a tilting object in $\text{CM}^{\mathbb{Z}}(\Lambda)$;*
- (2) *then there exists a triangle-equivalence $\mathcal{D}^b(KQ) \cong \text{CM}^{\mathbb{Z}}(\Lambda)$.*

Proof. (1) First, we have $e_F \Gamma_{\sigma} \cong \bigoplus_{a \in \sigma} M_a$. So we need to choose some degree shift of each M_a .

Suppose that all tagged arcs of σ are incident to the puncture. Suppose without loss of generality that they are tagged plain. We can lift σ to the set σ' of indecomposable objects of $\text{CM}^{\mathbb{Z}} \Lambda$ -modules of the form $(i, *)$ for $1 \leq i \leq n$. Let us prove that the graded module $T'_{\sigma'} = \bigoplus_{A \in \sigma'} A$ is tilting (it is T_{σ} if we forget the grading). Let us check that

$$\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}((i, *), \Omega^k(j, *)) = 0$$

for any $i, j \in \llbracket 1, n \rrbracket$ and $k \neq 0$. Thanks to Theorem 4.2, it is easy to compute projective covers of modules and we know that $\Omega^k(j, *) = (j + k(1 - n), *)$ if k is even and $\Omega^k(j, *) = (j + k(1 - n), \bowtie)$ if k is odd. Therefore, if k is odd, $\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}((i, *), \Omega^k(j, *)) = 0$.

Moreover, if $k \geq 2$, we get $j + k(1 - n) \leq j + 2 - 2n \leq 0 < i$. So

$$\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}((i, *), \Omega^k(j, *)) = 0.$$

If $k \leq -2$ is even, we want to prove that

$$\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}((i, *), \Omega^k(j, *)) = \text{Ext}_{\text{CM}^{\mathbb{Z}}(\Lambda)}^1((i, *), \Omega^{k+1}(j, *)) = 0.$$

We have $\Omega^{k+1}(j, *) = (j + (k+1)(1 - n), \bowtie)$ and $j + (k+1)(1 - n) \geq j + n - 1 \geq n \geq i$ and clearly $\text{Ext}_{\text{CM}^{\mathbb{Z}}(\Lambda)}^1((i, *), \Omega^{k+1}(j, *)) = 0$.

Let us now prove that $\text{thick}(T'_{\sigma'}) = \text{CM}^{\mathbb{Z}}(\Lambda)$. First of all, for any $i \in \mathbb{Z}$ such that $n \leq i < 2n - 2$, considering the short exact sequence

$$0 \rightarrow (i - n + 1, *) \rightarrow (i, i + 1) \oplus (n - 1, *) \rightarrow (i, 2n - 1) \rightarrow 0,$$

as $(i - n + 1, *)$ and $(n - 1, *)$ are in σ' and $(i, i + 1)$ is projective, we get that $(i, 2n - 1) \in \text{thick}(T'_{\sigma'})$. Now, for any $i \in \mathbb{Z}$ such that $n < i < 2n - 1$, using the short exact sequence

$$0 \rightarrow (n, 2n - 1) \rightarrow (i, 2n - 1) \oplus (n, *) \rightarrow (i, *) \rightarrow 0$$

we get that $(i, *) \in \text{thick}(T'_{\sigma'})$. Thus, as $\Omega^{2k}(j, *) = (j + 2k(1 - n), *)$, all the $(j, *)$ for $j \in \mathbb{Z}$ are in $\text{thick}(T'_{\sigma'})$. Consider $i, j \in \mathbb{Z}$ such that $1 < j - i < n$. We then have a short exact sequence

$$0 \rightarrow (i + 1 - n, *) \rightarrow (i, i + 1) \oplus (j - n, *) \rightarrow (i, j) \rightarrow 0$$

and, as $(i + 1 - n, *)$ and $(j - n, *)$ are in $\text{thick}(T'_{\sigma'})$ and $(i, i + 1)$ is projective, we get that $(i, j) \in \text{thick}(T'_{\sigma'})$. Finally, as $\Omega(i, *) = (i - n + 1, \infty)$, all the (i, ∞) are in $\text{thick}(T'_{\sigma'})$. We finished to prove in this case that $T'_{\sigma'}$ is tilting.

Suppose now that there is at least a tagged arc of σ which is not incident to the puncture. Thus, there exists a vertex i_0 of P such that i_0 does not have any incident internal edge in σ . Therefore, we can lift the tagged arcs of σ to a set σ' of indecomposable objects of $\text{CM}^{\mathbb{Z}}\Lambda$ -modules such that for any $A \in \sigma'$, $i_0 < A_1 < i_0 + n$ and $i_0 + 1 < A_2 < i_0 + 2n$. Let us prove that the graded module $T'_{\sigma'} = \bigoplus_{A \in \sigma'} A$ is tilting (it is T_{σ} if we forget the grading). Let us check that

$$\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}(A, \Omega^k B) = 0$$

for any $A, B \in \sigma'$ and $k \neq 0$. Let $B' = \Omega^k B$. Thanks to Theorem 4.2, $B'_1 = B_1 + k(1 - n)$ and $B'_2 = B_2 + k(1 - n)$.

Therefore, if $k > 0$, we get $B'_1 \leq B_1 + 1 - n \leq i_0 < A_1$. So $\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}(A, B') = 0$.

If $k < -1$, we want to prove that $\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}(A, B') = \text{Ext}^1_{\text{CM}^{\mathbb{Z}}(\Lambda)}(A, \Omega B') = 0$. If we denote $B'' = \Omega B'$, we have $B''_1 = B'_1 + 1 - n \geq B_1 + n - 1 \geq i_0 + n > A_1$. Then as the morphisms are directed positively, $\text{Ext}^1_{\text{CM}^{\mathbb{Z}}(\Lambda)}(A, \Omega B') = 0$.

For $k = -1$, by Theorem 3.16, we get

$$\text{Hom}_{\text{CM}^{\mathbb{Z}}(\Lambda)}(T'_{\sigma'}, \Omega^{-1}T'_{\sigma'}) \subset \text{Hom}_{\text{CM}\Lambda}(T_{\sigma}, \Omega^{-1}T_{\sigma}) = \text{Ext}^1_{\text{CM}\Lambda}(T_{\sigma}, T_{\sigma}) = 0.$$

Let us now prove that $\text{thick}(T'_{\sigma'}) = \text{CM}^{\mathbb{Z}}(\Lambda)$. Consider an indecomposable object $A \in \text{CM}^{\mathbb{Z}}\Lambda$ with $A_1 = i_0 + n$. Let $A' \in \text{CM}\Lambda$ be its image through the forgetful functor. It is a classical lemma about cluster tilting objects that there exists a short exact sequence

$$0 \rightarrow T'_1 \rightarrow T'_0 \rightarrow A' \rightarrow 0$$

of Cohen-Macaulay Λ -modules such that $T'_0, T'_1 \in \text{add}(T_{\sigma})$.

Let X' be an indecomposable summand of T'_1 . For any lift X of X' such that $\text{Ext}^1_{\text{CM}^{\mathbb{Z}}\Lambda}(A, X) \neq 0$, we have $i_0 < X_1 < i_0 + n$ by Theorem 4.2 (2), so such a lift X is unique and has to be in σ' . Moreover, in this case, $\text{Ext}^1_{\text{CM}^{\mathbb{Z}}\Lambda}(A, X) = \text{Ext}^1_{\text{CM}\Lambda}(A, X')$. Therefore, the unique lift T_1 of T'_1 which is in $\text{add}(T'_{\sigma'})$ satisfies $\text{Ext}^1_{\text{CM}^{\mathbb{Z}}\Lambda}(A, T_1) = \text{Ext}^1_{\text{CM}\Lambda}(A, T'_1)$ so we can lift the short exact sequence

$$0 \rightarrow T'_1 \rightarrow T'_0 \rightarrow A' \rightarrow 0$$

to a short exact sequence

$$0 \rightarrow T_1 \rightarrow T_0 \rightarrow A \rightarrow 0$$

of graded Cohen-Macaulay Λ -modules. As any indecomposable summand X of T_0 is between T_1 and A in the Auslander-Reiten quiver, we get $i_0 < X_1 \leq A_1 = i_0 + n$ so $X \in \sigma'$. Finally T_0 and T_1 are in $\text{add}(T'_{\sigma'})$ so $A \in \text{thick}(T'_{\sigma'})$.

For any $i \in \mathbb{Z}$ such that $i_0 + n < i < i_0 + 2n - 1$, there is a short exact sequence

$$0 \rightarrow (i_0 + n, i + 1) \rightarrow (i, i + 1) \oplus (i_0 + n, *) \rightarrow (i, *) \rightarrow 0$$

so, as $(i_0 + n, i + 1)$ and $(i_0 + n, *)$ are in $\text{thick}(T'_{\sigma'})$ and $(i, i + 1)$ is projective, $(i, *)$ is in $\text{thick}(T'_{\sigma'})$. As we already got the result for $(i_0 + n, *)$ and $(i_0 + n, \infty)$ and $\Omega^{-1}((i_0 + n, \infty)) = (i_0 + 2n - 1, *)$, all the $(i, *)$ for $i_0 + n \leq i \leq i_0 + 2n - 1$ are in $\text{thick}(T'_{\sigma'})$. Up to a shift by $1 - i_0 - n$, we already saw that these $(i, *)$ generate $\text{CM}^{\mathbb{Z}}(\Lambda)$. Therefore, $T'_{\sigma'}$ is a tilting object in $\text{CM}^{\mathbb{Z}}(\Lambda)$.

(2) Take the triangulation σ whose set of tagged arcs is

$$\{(P_1, P_3), (P_1, P_4), \dots, (P_1, P_n), (P_1, *), (P_1, \bowtie)\}.$$

The full subquiver Q of Q_σ with the set $Q_{\sigma,0} \setminus F$ of vertices is of type D_n . Thus, we have

$$\Gamma_\sigma^{\text{op}}/(e_F) \cong (KQ)^{\text{op}}.$$

Let σ' constructed from σ as before. Namely, $i_0 = 2$ and

$$\sigma' = \{(n+1, n+3), (n+1, n+4), \dots, (n+1, 2n), (n+1, *), (n+1, \bowtie)\}.$$

For any $A, B \in \sigma'$ and $k \in \mathbb{Z}$, we have $B(k)_1 = n+1 + kn$ so

$$\text{Hom}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}(A, B(k)) = 0$$

for $k \neq 0$. Indeed, if $k < 0$ this is immediate as morphisms go increasingly and if $k > 0$, $\text{Hom}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}(A, B(k)) = \text{Ext}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}^1(A, \Omega B(k))$. Moreover, $\Omega B(k)_1 = 2 + kn \geq n+2$ and for the same reason than before $\text{Ext}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}^1(A, \Omega B(k)) = 0$.

Thus, by Theorem 3.16, we have the following isomorphism

$$\begin{aligned} \text{End}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}(e_F \Gamma_\sigma) &\cong \text{End}_{\underline{\text{CM}} \Lambda}(e_F \Gamma_\sigma) \\ &\cong \Gamma_\sigma^{\text{op}}/(e_F). \end{aligned}$$

Because $\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)$ is an algebraic triangulated Krull-Schmidt category, and $e_F \Gamma_\sigma$ is a tilting object in $\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)$, by Theorem 4.4, there exists a triangle-equivalence

$$\underline{\text{CM}}^{\mathbb{Z}}(\Lambda) \cong \mathcal{K}^b \left(\text{proj End}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}^{\text{op}}(e_F \Gamma_\sigma) \right).$$

Since $\text{gl. dim } KQ < \infty$, we have a triangle-equivalence

$$\mathcal{K}^b \left(\text{proj End}_{\underline{\text{CM}}^{\mathbb{Z}}(\Lambda)}^{\text{op}}(e_F \Gamma_\sigma) \right) \cong \mathcal{K}^b(\text{proj } KQ) \cong \mathcal{D}^b(KQ).$$

Therefore there is a triangle-equivalence $\underline{\text{CM}}^{\mathbb{Z}}(\Lambda) \cong \mathcal{D}^b(KQ)$. \square

REFERENCES

- [1] Claire Amiot, Osamu Iyama, and Idun Reiten. Stable categories of Cohen-Macaulay modules and cluster categories. *Amer. J. Math.*, 137(3):813–857, 2015.
- [2] Tokuji Araya. Exceptional sequences over graded Cohen-Macaulay rings. *Math. J. Okayama Univ.*, 41:81–102 (2001), 1999.
- [3] M. Auslander. Functors and morphisms determined by objects. In *Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976)*, pages 1–244. Lecture Notes in Pure Appl. Math., Vol. 37. Dekker, New York, 1978.
- [4] A. I. Bondal and M. M. Kapranov. Framed triangulated categories. *Mat. Sb.*, 181(5):669–683, 1990.
- [5] A. B. Buan, O. Iyama, I. Reiten, and D. Smith. Mutation of cluster-tilting objects and potentials. *Amer. J. Math.*, 133(4):835–887, 2011.
- [6] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov. Tilting theory and cluster combinatorics. *Adv. Math.*, 204(2):572–618, 2006.
- [7] Giovanni Cerulli Irelli and Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces, Part III: tagged triangulations and cluster monomials. *Compos. Math.*, 148(6):1833–1866, 2012.
- [8] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I*. John Wiley & Sons Inc., New York, 1981. With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication.
- [9] L. de Thanhoffer de Völcsey and M. Van den Bergh. Explicit models for some stable categories of maximal Cohen-Macaulay modules, June 2010. arXiv: 1006.2021.
- [10] L. Demonet and X. Luo. Ice quivers with potentials associated with triangulations and Cohen-Macaulay modules over orders, July 2013. arXiv: 1307.0676, accepted for publication to Trans. Amer. Math. Soc.
- [11] H. Derksen, J. Weyman, and A. Zelevinsky. Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)*, 14(1):59–119, 2008.
- [12] S. Fomin, M. Shapiro, and D. Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

- [13] C. Geiß, B. Leclerc, and J. Schröer. Kac-Moody groups and cluster algebras. *Adv. Math.*, 228(1):329–433, 2011.
- [14] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.
- [15] D. Happel. Auslander-Reiten triangles in derived categories of finite-dimensional algebras. *Proc. Amer. Math. Soc.*, 112(3):641–648, 1991.
- [16] H. Hijikata and K. Nishida. Classification of Bass orders. *J. Reine Angew. Math.*, 431:191–220, 1992.
- [17] Hiroaki Hijikata and Kenji Nishida. Primary orders of finite representation type. *J. Algebra*, 192(2):592–640, 1997.
- [18] O. Iyama. Representation theory of orders. In *Algebra—representation theory (Constanta, 2000)*, volume 28 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 63–96. Kluwer Acad. Publ., Dordrecht, 2001.
- [19] O. Iyama and B. Lerner. Tilting bundles on orders on P^d , June 2013. arXiv: 1306.5867.
- [20] O. Iyama and R. Takahashi. Tilting and cluster tilting for quotient singularities. *Math. Ann.*, 356(3):1065–1105, 2013.
- [21] Hiroshige Kajiura, Kyoji Saito, and Atsushi Takahashi. Matrix factorization and representations of quivers. II. Type ADE case. *Adv. Math.*, 211(1):327–362, 2007.
- [22] Hiroshige Kajiura, Kyoji Saito, and Atsushi Takahashi. Triangulated categories of matrix factorizations for regular systems of weights with $\varepsilon = -1$. *Adv. Math.*, 220(5):1602–1654, 2009.
- [23] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.
- [24] B. Keller and I. Reiten. Acyclic Calabi-Yau categories. *Compos. Math.*, 144(5):1332–1348, 2008. With an appendix by Michel Van den Bergh.
- [25] D. Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc. (3)*, 98(3):797–839, 2009.
- [26] D. Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions, June 2012. arXiv: 1206.1798.
- [27] Sefi Ladkani. On Jacobian algebras from closed surfaces, July 2012. arXiv: 1207.3778.
- [28] K. W. Roggenkamp. The construction of almost split sequences for integral group rings and orders. *Comm. Algebra*, 5(13):1363–1373, 1977.
- [29] K. W. Roggenkamp and J. W. Schmidt. Almost split sequences for integral group rings and orders. *Comm. Algebra*, 4(10):893–917, 1976.
- [30] D. Simson. *Linear representations of partially ordered sets and vector space categories*, volume 4 of *Algebra, Logic and Applications*. Gordon and Breach Science Publishers, Montreux, 1992.
- [31] Y. Yoshino. *Cohen-Macaulay modules over Cohen-Macaulay rings*, volume 146 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, 464-8602 NAGOYA, JAPAN

E-mail address: Laurent.Demonet@normalesup.org

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, 464-8602 NAGOYA, JAPAN

E-mail address: luoxueyu2012@gmail.com